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HIGH VERSUS LOW ANTENNAE IN RADIO TELEGRAPHY AND TELEPHONY

BY
EDWARD BENNETT
Professor of Electrical Engineering
The University of Wisconsin

THE UNIVERSITY OF WISCONSIN
ENGINEERING EXPERIMENT STATION

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HIGH VERSUS LOW ANTENNAE IN RADIO TELEGRAPHY AND TELEPHONY

I. PREFATORY CONSIDERATIONS

1. Current Notions Relating to the Height of Radio Antennae

In a paper presented before the British Institution of Electrical Engineers in 1899, Marconi announced a relation between the working telegraphic distance of a pair of wireless stations and the height of the station antennae. This relation, which has come to be known as Marconi's law, is as follows: For stations with antennae of equal height, "the distance at which signals can be obtained varies approximately with the square of the distance of the capacities from earth, or perhaps with the square of the length of the vertical conductors."*

This relation, which is based upon experiments between stations with antennae each consisting of a single vertical wire or a single wire connected to a capacity area of very moderate dimensions, has exerted and now exerts a guiding or dominating influence in the practice of wireless telegraphy. The great elevation—from 100 to 1000 feet—at which the capacity areas are mounted in all wireless stations is conclusive evidence either of the necessity or of the importance of high antennae in the minds of those practicing the art of wireless telegraphy at the present time.

From the very early years of the art, the practice in the construction of wireless antennae has been to use, not single wires, but a multiplicity of wires arranged in the form of a fan, a harp, an umbrella, a cylindrical cage, or an inverted cone or pyramid. These wires constitute an extended "capacity area," and, as previously stated, the practice is

*Marconi, *Wireless Telegraphy—Jour. of the Inst. of Electrical Engineers* 1899 Vol. 28, page 279.

to mount this extended capacity area, or at least a large part of it, at a great elevation above the earth. For example, the capacity area of the government station at Arlington, D. C., consists of three approximately horizontal wire harps suspended at a mean elevation of about 500 feet above the surface of the earth. Each harp contains 23 strings and has a width of 88 feet and a mean length of 300 ft.

The following elementary considerations seemed to the writer to warrant a critical examination of the practice of mounting such extended capacity areas at these great elevations.

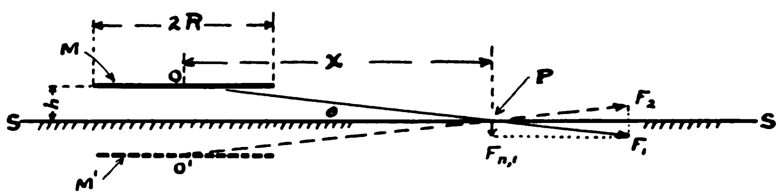

FIG. 1

2. THE ELECTROSTATIC FIELD AT A GREAT DISTANCE FROM AN EXTENDED CHARGED SHEET

In Fig. 1, let M represent an extended circular sheet of conducting material insulated from and parallel to the surface of the earth. Imagine this conducting or capacity area M to be maintained at a steady voltage E above the potential of the earth, and let us calculate the potential gradient which is thereby set up at a point P near the surface of the earth and at a great distance from the capacity area.

Let E represent the difference of potential in volts between the area M and the earth.
Let x represent the horizontal distance from the center of M to the point P.
Let R represent the radius of the area M.
Let h represent the height of M above the earth.
Let p represent the permittivity of the air (8.84×10^{-14})
All distances are to be expressed in centimeters.

Suppose now that h is small in comparison with R and that the distance x in comparison with R is very large. For example, suppose the radius R is of the order of 60 meters (200 ft.), that the height h is between 1 meter and 30 meters (3 and 100 feet), and that the distance x to the point P is 10 kilometers (6.2 miles) or more. For these proportions the following statements are approximately correct.

Neglecting "edge effects" and displacement from the upper face of M, the capacity C of the sheet M with reference to the earth is

$$C = p\frac{\pi R^2}{h}\text{farads}$$

The quantity of electricity (Q) on M is

$$Q = CE = p\frac{\pi R^2}{h}E \text{ coulombs}$$

The potential gradient F_1 at P due to the charge Q on M is exerted in the direction OP and is

$$F_1 = \frac{1}{4\pi p}\,\frac{Q}{(h^2+x^2)} = \frac{R^2E}{4h(h^2+x^2)} \text{ volts per cm.}$$

Since $h^2 \lessdot .00001\ x^2$,

$$F_1 = \frac{R^2E}{4hx^2} \text{ volts per cm. (approximately)}$$

The gradient at P which results from the distribution of the charge $(-Q)$ on the surface of the earth is calculated by introducing the electrical image of the charge Q with reference to the equipotential surface SS, which is hereinafter treated as a plane surface. The image of Q is the charge $(-Q)$ located on the surface M′ at the distance h below the plane SS.

The charge $(-Q)$ at M′ would give rise at the point P to a gradient F_2 exerted along the line O′P and equal to

$$F_2 = -\frac{R^2E}{4hx^2} \text{ volts per cm.}$$

The gradients F_1 and F_2 at P may be resolved into components parallel to and normal to the surface of the earth. The components parallel to the surface neutralize, and those normal to the surface add. The component of F_1 normal to the surface of the earth is—

$$F_{n1} = F_1 \sin \theta = F_1 \frac{h}{\sqrt{h^2 + x^2}} = F_1 \frac{h}{x}$$

$$F_{n1} = \frac{R^2E}{4x^3} \text{ volts per cm.}$$

Therefore the resultant potential gradient F at the point P is normal to the surface of the plane SS, is directed downward, and is approximately expressed by

$$F = \frac{R^2E}{2x^3} \text{ volts per cm.}$$

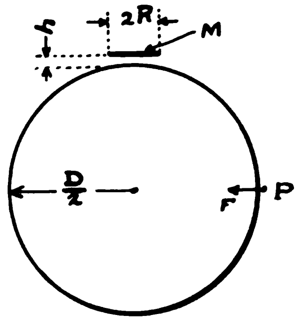

FIG 2

The expression for the gradient at P does not involve h, the height of capacity area M. Therefore, *within the limits previously specified*, the gradient at P due to an extended sheet M maintained at a given potential E above the ground is independent of the height of the sheet above the ground.*

* It is to be recognized that this treatment is not rigorous and that the conclusions apply only within certain limits. If h is made large as compared with R, the capacity of the sheet M to the plane SS becomes independent of h and equal to 8pR farads. Therefore the gradient at P is normal to the surface of the plane and is equal to $\frac{4Rh\,E}{\pi x^3}$ volts per cm. That is, the gradient at P will be directly porportional to the height of the sheet M above the surface SS.

The fact that the earth's surface is a spherical surface and not a plane surface does not alter the conclusion that the gradient at the point P is independent of the height of the sheet above the ground. For example, by applying the method of images to the spherical surface shown in Fig. 2, it may be shown that the gradient at the point P which is at a quadrant's distance from M is given by the expression

$$F = \frac{\sqrt{2}\, R^2 E}{D^3} \text{ volts per cm. (approximately)}$$

This expression does not involve h, the height of the sheet M above the surface of the sphere. (As in the case of the plane surface, this expression applies only for the case in which h is small in comparison with R.)

In view of the fact that the height of the capacity area is (within limits) without influence upon the steady state of the medium at P, this question now arises. If the potential of the sheet M, instead of being maintained constant, is caused to vary in a periodic manner, how will the magnitude of the disturbance thereby set up in the medium at the point P be affected by the height of the sheet M?

3. Simplification of the Radiating System

To answer the question thus raised by the discussion of the steady state of the field requires the application of the equations of the electro magnetic field to the radiating system shown in Fig. 3. In its simplest form the radiating system

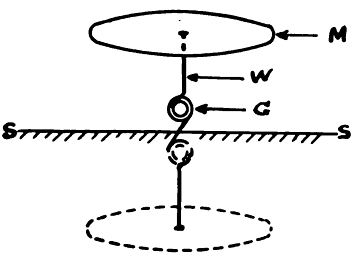

Fig. 4

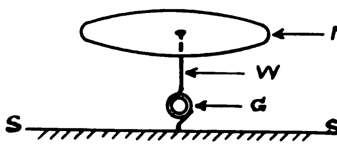

Fig. 3

comprises an extended circular plate M with its surface parallel to the surface of the earth, a vertical conductor W and a generator G generating a sine e. m. f. To treat such a system, the surface SS is imagined to have infinite conductivity. It thus becomes an equipotential surface and the effect of the distributions of current and charge over this surface is determined by replacing this conducting surface by the images of elements M, W, and G in the surface, as in Fig. 4.

The conditions to be fulfilled at the boundaries of Fig. 4—that is, at the surfaces of the conductors—are so involved that a rigorous analytical treatment is impossible. If the conditions are simplified by assuming the conducting sheet M and wire W to have infinite conductivity, the boundary conditions are still too involved for treatment. It becomes necessary, therefore, to further simplify the radiating system by depicting it as in Fig. 5. In Fig. 5 a positive charge Q distributed over a circular area of radius R is assumed to move up and down in such a manner that at any moment its elevation (h) above the surface SS is given by the expression

$$h = h_0 \cos \omega t.$$

A negative charge $(-Q)$, also distributed over a circular area of radius R, moves up and down so that its elevation is given by the expression

$$h = -h_0 \cos \omega t$$

The charges Q and $-Q$ are not uniformly distributed over the two circular areas, but they may be imagined to be confined

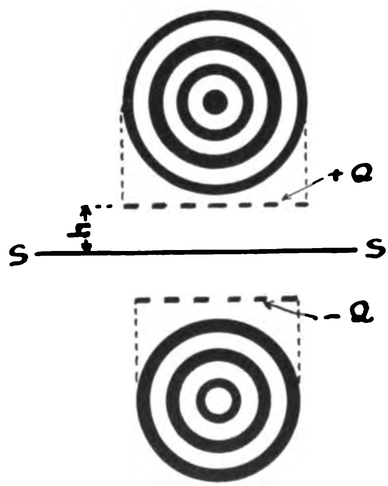

FIG. 5

to insulated circular strips. When the two charges move up and down, the circular strips carrying the positive charge may be imagined to pass between the circular strips carrying the negative charge at the instant both charges pass in opposite directions across the surface SS. The steady states at great distances which correspond to the instantaneous states of such a radiating system are identically the same as the states corresponding to a Fig. 4 system, provided the total voltage generated in the two generators of Fig. 4 is given by the expression

$$e = 2\, E \cos \omega t = \frac{2Q h_0}{\pi R^2 p} \cos \omega t$$

In the case of the radiating system depicted in Fig. 5, there are no conduction currents to deal with, and the electric charges move in a simple predetermined manner. We proceed (a) to set up the differential equations applying to this system, (b) to indicate a solution of these equations, and (c) from this solution to draw conclusions as to the relative merits of high versus low capacity areas in wireless telegraphy.

II. DIFFERENTIAL EQUATIONS OF THE ELECTROMAGNETIC FIELD..

4. Notation.

At any point in the electromagnetic field,

Let

ρ represent the volume density of electricity in coulombs per cu. cm.

V " the velocity of the moving charge in cm. per sec.

F " the electric force or potential gradient in volts per cm.

H " the magnetic force in ampere-turns per cm.

D " the electrostatic flux density or displacement in coulombs per sq. cm.

B " the magnetic flux density in webers per sq. cm.

Φ " the retarded displacement potential.

A " the retarded vector potentiial.

Let

f represent the frequency of the radiating system in cycles per sec.

p " the permittivity of the medium in coulombs per sq. cm. per volt per cm.

For free space $p = \frac{1}{4\pi 9\ 10^{11}}$ or $8.84\ 10^{-14}$

μ " the permeability of the medium in webers per sq. cm. per ampere-turn per cm.

For free space $\mu = \frac{4\pi}{10^9}$ or 1.257 10^{-8}

s " $\frac{1}{\sqrt{\mu p}} = 3\ 10^{10} =$ velocity of light.

∇^2 " $\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$

The quantities represented by bold faced capitals are vector quantities. Let their X, Y, and Z components be designated by the subscripts $_1$, $_2$, and $_3$. Thus,

Let

D_1 D_2 D_3 represent the X, Y, and Z components of the displacement.

V_1 V_2 V_3 represent the X, Y, and Z components of the velocity, etc.

It will be noted that all quantities are to be expressed in terms of the Ampere, Ohm, Ampere-turn, Weber system of units.

5. Fundamental Relations Expressed in Vector Notation

The fundamental relations which must be satisfied at all points of the electromagnetic field are expressed by the following differential equations.

$$\text{div } \mathbf{D} = \rho \quad (1)$$

$$\text{div } \mathbf{B} = 0 \quad (2)$$

$$\text{curl } \mathbf{H} = \frac{d\mathbf{D}}{dt} + \rho\mathbf{V} \quad (3)$$

$$\text{curl } \mathbf{F} = -\frac{d\mathbf{B}}{dt} \quad (4)$$

$$\mathbf{D} = p\mathbf{F} \quad (5)$$

$$\mathbf{B} = \mu\mathbf{H} \quad (6)$$

These same relations when expressed by differential equations involving the rectangular components of the vectors take the following forms.

6. Fundamental Relations Expressed in Rectangular Coordinates

Equations (1) and (2) may be written:

$$\frac{dD_1}{dx} + \frac{dD_2}{dy} + \frac{dD_3}{dz} = \rho \tag{1a}$$

$$\frac{dB_1}{dx} + \frac{dB_2}{dy} + \frac{dB_3}{dz} = 0 \tag{2a}$$

Equation (3) may be written:

$$\mathbf{i}\left(\frac{dH_3}{dy} - \frac{dH_2}{dz}\right) + \mathbf{j}\left(\frac{dH_1}{dz} - \frac{dH_3}{dx}\right) + \mathbf{k}\left(\frac{dH_2}{dx} - \frac{dH_1}{dy}\right) =$$

$$\mathbf{i}\left(\frac{dD_1}{dt} + \rho V_1\right) + \mathbf{j}\left(\frac{dD_2}{dt} + \rho V_2\right) + \mathbf{k}\left(\frac{dD_3}{dt} + \rho V_3\right)$$

This yields the three equations:

$$\frac{dH_3}{dy} - \frac{dH_2}{dz} = \frac{dD_1}{dt} + \rho V_1 \tag{3a}$$

$$\frac{dH_1}{dz} - \frac{dH_3}{dx} = \frac{dD_2}{dt} + \rho V_2 \tag{3b}$$

$$\frac{dH_2}{dx} - \frac{dH_1}{dy} = \frac{dD_3}{dt} + \rho V_3 \tag{3c}$$

In like manner, equation (4) yields the following three equations:

$$\frac{dF_3}{dy} - \frac{dF_2}{dz} = -\frac{dB_1}{dt} \tag{4a}$$

$$\frac{dF_1}{dz} - \frac{dF_3}{dx} = -\frac{dB_2}{dt} \tag{4b}$$

$$\frac{dF_2}{dx} - \frac{dF_1}{dy} = -\frac{dB_3}{dt} \tag{4c}$$

7. Physical Interpretation of the Differential Equations

The relations expressed in the differential equations 1 to 4 are more familiar to engineers when stated in a form more suitable for application to circuits of finite dimensions. It may, therefore, be well, before we proceed to the solution of these differential equations, to identify the equations with the more familiar statements of the laws they express.

Equation 4 results from the application of Faraday's Law of Induction to a circuit of infinitesimal dimensions. The law of induction is, "The electromotive force induced in a closed circuit, or the line integral of the electric force around the circuit, is equal to the rate of decrease of the magnetic flux threading the circuit." This is called by Heaviside the second law of circuitation. Consider the application of this law to the small circuit bounding the infinitesimal square parallel to the XY plane in Fig. 6.

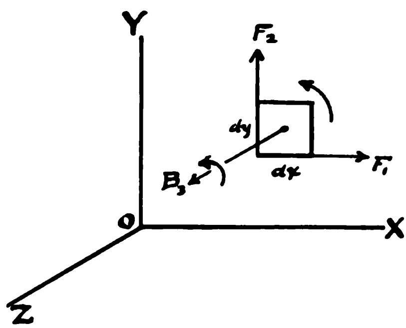

FIG. 6

The magnetic flux threading this circuit is B_3dxdy

The rate of decrease of this flux is $-\frac{dB_3}{dt}$ dxdy

Now if F_1 and F_2 represent the X and Y components of the electric force or voltage gradient at this point, it is evident that the resultant or net electromotive force around the circuit in the direction indicated (or, in other words, the line integral of the electric force around the boundary of the square) is given by the expression

$$\text{Line integral of } \mathbf{F} = F_1\,dx + \left[F_2 + \frac{dF_2}{dx}dx\right]dy - \left[F_1 + \frac{dF_1}{dy}dy\right]dx - F_2dy = \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right)dx\,dy$$

According to Faraday's law, the electromotive force induced in a circuit around the boundary of this square is equal to the rate of decrease of the flux threading the square.

Therefore the line integral of the electric force **F** around the area dydx or—

$$\left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) dx\, dy \text{ is equal to } - \frac{dB_3}{dt} dx\, dy$$

Whence

$$\frac{\text{Line integral of } \mathbf{F} \text{ around } dx\, dy}{\text{area}} \text{ or } \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) = - \frac{dB_3}{dt}$$

This is equation (4c). In like manner equations (4a) and (4b) may be derived.

Now the *curl of a vector* **F** at any point P *and in any plane* passing through that point is defined as a vector **M** whose length is equal to the line integral of the vector **F** taken around the boundary of an infinitesimal portion of the plane divided by the area of the infinitesimal portion. The vector **M** is to be drawn normal to the plane and in that direction in which a right hand screw would advance if it were threaded through the plane and rotated in the direction in which the boundary was traversed in taking the line integral. At the given point P there will be some plane for which this quotient, or the curl, has a maximum value. This maximum value is called "*The* Curl of the vector **F** at the point P." If the curl of the vector **F** in three planes parallel to the XY, XZ, and YZ planes is taken, the three vectors so obtained are the Z, Y, and X components of "*The* Curl of the vector."

To summarize the above discussion, the quotient obtained by dividing the line integral of the electric force **F** around the boundary of a small area parallel to the XY plane by the area was found to be $\left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right)$. Through the application of Faraday's Law of Induction, this quotient was shown to equal the rate of decrease of the Z component of the flux density at the point. In other words, the curl of the electric force **F** in a plane parallel to the XY plane, or the Z component of the curl of **F**, is equal to the rate of decrease of the Z component of the flux density. Likewise, the X and Y components of the curl of **F** are equal respectively to the rates of decrease of the X and Y components of the flux density B. Or

$$\text{curl}\,\mathbf{F} = -\frac{d\mathbf{B}}{dt} \qquad (4)$$

8. Equation (3) is obtained by applying to a circuit of infinitesimal dimensions the familiar conception that the

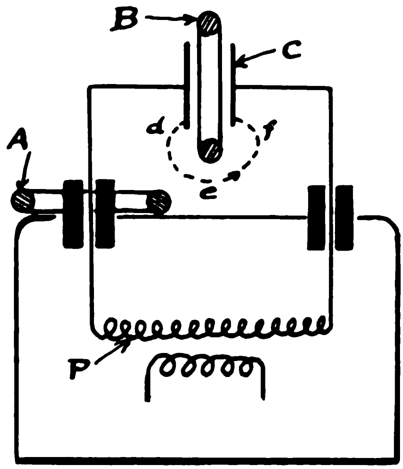

FIG. 7

magnetomotive force in ampere turns exerted around any complete circuit, or the line integral of the magnetic force around the circuit, is equal to the current passing through and looping with the circuit around which the line integral is taken. By the current passing through the circuit is meant the sum of the conduction current plus the convection current plus the displacement current. This is called by Heaviside the first law of circuitation.

To illustrate a specific application of this law to a circuit of finite dimensions, suppose we wish to determine the magnetomotive force exerted upon the magnetic circuit—the iron core—of the current transformer illustrated in two different positions A and B in Fig. 7.

To find the current passing through and looped with the core we imagine any surface, plane or curved, of which the core is the boundary. The current looping with the iron core at any instant is, then, the net current passing across this surface. Imagine a plane surface of which the core is the boundary. Then with the current transformer in the position A, substantially the only current which crosses the plane surface is the conduction current in the high tension lead of the power transformer P. (The secondary circuit of the current transformer is assumed to be open.)

Suppose, however, the current transformer is shifted to the position B, a position in which the plane surface bounded by the core cuts through the dielectric of the condenser C, and consequently, a position in which no conduction current crosses the plane surface. In this position the current crossing the plane surface is a displacement current. This displacement current—the rate of change of the electrotrostatic flux which passes across the plane surface bounded by the core—is less than the conduction current in the transformer lead at A by the displacement which takes place between the leads along paths, as *d e f*, which do not loop through the iron core. If the leads are short and the condenser plates large, the magnetomotive force exerted upon the core in the position B will be only slightly lower than in the position A.

As in the previous case, let these considerations be applied to a small square circuit of infinitesimal dimensions similar to that shown in Fig. 6.

The line integral of the magnetic force H around the boundary of the square is—

$$\left(\frac{dH_2}{dx} - \frac{dH_1}{dy}\right) dx\ dy$$

The current passing through the area dxdy is—

$$\left(\rho V_3 + \frac{dD_3}{dt}\right) dx\ dy$$

Since the magnetomotive force around the boundary of the area equals the current through the area, these expressions are equal.

$$\text{Whence } \left(\frac{dH_2}{dx} - \frac{dH_1}{dy}\right) dx\, dy = \left(\rho V_3 + \frac{dD_3}{dt}\right) dx\, dy$$

$$\text{or } \left(\frac{dH_2}{dx} - \frac{dH_1}{dy}\right) = \left(\rho V_3 + \frac{dD_3}{dt}\right)$$

This is equation (3c). The left member is the curl of the magnetic force **H** in a plane parallel to the XY plane, or it is the Z component of the curl of **H**. The right member is the expression for the Z component of the convection current density plus the Z component of the displacement current density.

Equations (3a) and (3b) may be derived in a similar manner.

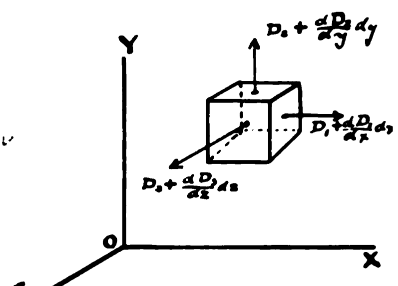

FIG. 8

9. Equation (1) expresses the fact that the Faraday tubes of electric force originate on the electric charges. The relation is perhaps more familiar in either of the following forms:

"The number of tubes of displacement which cross any closed surface in the field is equal to the quantity of electricity contained within the surface," or, "The surface integral of the displacement taken over any closed surface is equal to the quantity of electricity contained within the surface."

Let this law be applied to the small cubical volume shown in Fig. 8.

The surface integral of the displacement taken over the surface of the cube is:

$$-D_1\, dydz + \left[D_1 + \frac{dD_1}{dx}dx\right] dy\, dz - D_2\, dx\, dz$$

$$+\left[D_2 + \frac{dD_2}{dy}dy\right] dxdz - D_3\, dxdy + \left[D_3 + \frac{dD_3}{dz}dz\right] dxdy$$

$$\text{or } \left[\frac{dD_1}{dx} + \frac{dD_2}{dy} + \frac{dD_3}{dz}\right] dxdydz$$

The quantity of electricity within the cube is ρ dxdydz

$$\text{Whence}\left[\frac{dD_1}{dx}+\frac{dD_2}{dy}+\frac{dD_3}{dz}\right] dxdydz = \rho\ dxdydz$$

$$\text{or}\left[\frac{dD_1}{dx}+\frac{dD_2}{dy}+\frac{dD_3}{dz}\right] = \rho \qquad (1a)$$

The left member of equation (1a) is the quotient obtained by dividing the surface integral of D taken over the surface of the infinitesimal cube by the volume of the cube. The value of this quotient is called the divergence of the vector D at the point.

10. Equation 2 expresses the fact that the tubes of magnetic induction do not originate or diverge from any portion of space. They are closed tubes, linking with the current and returning into themselves.

III. INTEGRATION OF THE DIFFERENTIAL EQUATIONS

11. To Obtain the Differential Equations in a Form Involving Only One Dependent Variable, as D_1

Differentiating (3a) with respect to (t),

$$\frac{d^2 H_3}{dt\,dy}-\frac{d^2 H_2}{dt\,dz}=\frac{d^2 D_1}{dt^2}+\frac{d}{dt}(\rho V_1) \qquad (7)$$

Substituting in (4b) and (4c) μH for B, differentiating (4c) with respect to y, and (4b) with respect to z, and substituting the values so obtained for $\frac{d^2 H_3}{dt\,dy}$ and $\frac{d^2 H_2}{dt\,dz}$ in equation (7),

$$\frac{1}{\mu}\left[\frac{d^2 F_1}{dy^2}-\frac{d^2 D_2}{dy\,dx}+\frac{d^2 F_1}{dz^2}-\frac{d^2 F_3}{dz\,dx}\right]=\frac{d^2 D_1}{dt^2}+\frac{d}{dt}(\rho V_1) \qquad 8$$

Substituting for F in equation (8), its value from equation (5),

$$\frac{1}{\mu p}\left[\frac{d^2 D_1}{dy^2}-\frac{d^2 D_2}{dy\,dx}+\frac{d^2 D_1}{dz^2}-\frac{d^2 D_3}{dz\,dx}\right]=\frac{d^2 D_1}{dt^2}+\frac{d}{dt}(\rho V_1) \qquad 9$$

Differentiating (1a) with respect to x and substituting the value so obtained for $\left(-\frac{d^2 D_2}{dy\,dx} - \frac{d^2 D_3}{dz\,dx}\right)$ in equation (9)

$$\left[\frac{d^2 D_1}{dx^2} + \frac{d^2 D_1}{dy^2} + \frac{d^2 D_1}{dz^2}\right] - \mu p \frac{d^2 D_1}{dt^2} = \frac{d\rho}{dx} + \mu p \frac{d}{dt}(\rho V_1) \quad (10a)$$

Writing s for $\frac{1}{\sqrt{\mu p}} = 3\ 10^{10}$

and ∇^2 for $\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$

Equation (10a) may be written,

$$\nabla^2 D_1 - \frac{1}{s^2}\frac{d^2 D_1}{dt^2} = \frac{d\rho}{dx} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1) \quad (10a)$$

$$\text{or} \left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_1 = \frac{d\rho}{dx} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1) \quad (10a)$$

In like manner the following equations may be obtained,

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_1 = \frac{d\rho}{dy} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1) \quad (10b)$$

$$\text{and} \left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_3 = \frac{d\rho}{dz} + \frac{1}{s^2}\frac{d}{dt}(\rho V_3) \quad (10c)$$

12. To Obtain Differential Equations Involving H_1 Only

Differentiating (4a) with respect to (t), and substituting for B_1 its value μH_1,

$$\frac{d^2 F_3}{dt\,dy} - \frac{d^2 F_2}{dt\,dz} = -\mu\frac{d^2 H_1}{dt^2} \quad (11)$$

Substituting for D in equations (3b) and (3c) its value pF, differentiating (3c) with respect to (y) and (3b) with respect to (z), and substituting the values so obtained for $\frac{d^2 F_3}{dt\,dy}$ and $\frac{d^2 F_2}{dt\,dz}$ in equation (11),

$$\frac{1}{p}\left[\frac{d^2 H_2}{dx\,dy} - \frac{d^2 H_1}{dy^2} - \frac{d}{dy}(\rho V_3) - \frac{d^2 H_1}{dz^2} + \frac{d^2 H_3}{dz\,dx} + \frac{d}{dz}(\rho V_2)\right]$$

$$= -\mu\frac{d^2 H_1}{dt^2} \quad (12)$$

Differentiating (2a) with respect to (x) and substituting the value so obtained for $\left(\frac{d^2 H_2}{dx\,dy} + \frac{d^2 H_3}{dz\,dx}\right)$ in equation (12),

$$\left[\frac{d^2 H_1}{dx^2} + \frac{d^2 H_1}{dy^2} + \frac{d^2 H_1}{dz^2}\right] - \mu p \frac{d^2 H_1}{dt^2} = -\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right] \qquad (13a)$$

This may be written:

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] H_1 = -\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right] \qquad (13a)$$

In like manner the following equations may be obtained:

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] H_2 = -\left[\frac{d}{dz}(\rho V_1) - \frac{d}{dx}(\rho V_3)\right] \qquad (13b)$$

and $$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] H_3 = -\left[\frac{d}{dx}(\rho V_2) - \frac{d}{dy}(\rho V_1)\right] \qquad (13c)$$

13. The Dalembertian Operator

The differential equations for which the solution is desired take the forms:

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_1 = \frac{d\rho}{dx} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1) \qquad (10a)$$

and $$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] H_1 = -\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right] \qquad (13a)$$

That is to say, in the solutions or integrated equations, D_1 and H_1 must be such functions of time and of the position of a point in space that the operation $\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]$ applied to the function will yield a result whose value is determined by the volume density (ρ) and the current density ($\rho\mathbf{V}$) at the point.

It is very convenient to have a name for the result of the operation $\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]$ upon a quantity. H. A. Lorentz has suggested* that the result of the operation be

* H. A. Lorentz, *Theory of Electrons*, page 17.

called the "Dalembertian of the quantity," since d'Alembert was the first to solve the differential wave equation involving the operation $\left[\frac{d^2}{dx^2} - \frac{1}{s^2}\frac{d^2}{dt^2}\right]$ which is a special case of $\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]$.

Adopting this suggestion, we may say that the Dalembertians of the components of **D** and **H** are given in equations (10) and (13) in terms of the volume density (ρ) and current density ($\rho\mathbf{V}$).

Now in the system of moving charges depicted in Fig. 5, (ρ) and ($\rho\mathbf{V}$) are known functions of time and of the position of a point in space. That is to say, the right hand members of equations (10a) and (13a) are functions of known form, and the problem is to find the functional expression which will give the value of D_1, or of H_1, at any point in space and for any moment of time. If the known form assumed by the right hand member of equation (10a) is represented by the expression f (t, X, Y, Z), we have given—

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_1 = f\,(t, X, Y, Z) \qquad (14)$$

Or, dropping the subscripts, the problem is—
Given, Dalembertian D = f (t, X, Y, Z),
to find, the expression for D.

We proceed to demonstrate the following proposition.

14. Proposition That the "Retarded Potentializing Operation" Performed Upon the f (t, x, y, z) Yields a Function Whose Dalembertian Equals the f (t, x, y, z).*

"If the Dalembertian of D equals f (t, X, Y, Z), then the value of D may be found by an operation which may be called the operation of forming the "retarded potential" of the f (t, X, Y, Z). The operation of forming the retarded potential of the f (t, X, Y, Z) may be symbolized by stating that the value of D will be given by the equation,

* Whittaker, *History of the Theories of Aether and Electricity*, pages 268 & 298.
H. A. Lorentz, *The Theory of Electrons*, page 233.
Ludwig Lorenz, *Phil. Mag.* (1867) Vol. 34, page 287.
M. B. Riemann, *Phil. Mag.* (1867) Vol. 34, page 368.

$$D = -\frac{1}{4\pi}\int^{\text{all space}} \frac{f\left[\left(t-\frac{r}{s}\right), X,Y,Z,\right]}{r} dv \qquad (15)$$

This expression is to be read as follows: The value of D at a given instant (t) and for a given point $P \equiv (x, y, z)$ is equal to $\left(-\frac{1}{4\pi}\right)$ times the summation obtained—

(a) by dividing all space into volume elements (dv),

(b) dividing the volume of each element by its distance $r = \sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}$ from the point P.

(c) multiplying this quotient by the value of the f (t, X, Y, Z) at the volume element, *not for the instant* (*t*), but for the instant of time $\left(t - \frac{r}{s}\right)$, or r/s seconds earlier,

(d) and finally summing up all the products so obtained."

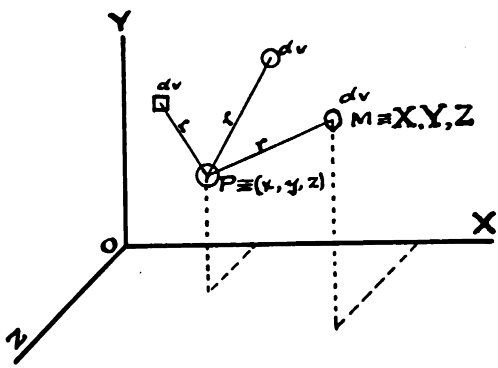

FIG. 9

Before taking up the proof of this proposition it may be noted that if there are no moving charges in the field the operation of obtaining the retarded potential becomes identically the same as the more familiar operation of obtaining the potentials in the gravitational or in the electrostatic field. If there are no moving charges, the Dalembertian of D degrades to the Laplacian of D.

15. Proof for Charges at a Distance from the Point

To show that the value of D defined by equation (15) satisfies the differential relation expressed in equation (14).

Let P ≡ (x, y, z), Fig. 9, be any point for which the value of D is desired for the instant (t). In performing the integration indicated in equation (15), we are concerned with only those volume elements for which the f (t, X, Y, Z) has a value at the instant $\left(t - \frac{r}{s}\right)$. That is, only those volume elements which contain a charge at the instant $\left(t - \frac{r}{s}\right)$ contribute to the integral. All the balance of space contributes nothing to the integral. A few of the volume elements presumed to contribute to the integral have been shown in Fig. 9.

Consider first the part of the integral contributed by any volume element whatsoever except the element immediately surrounding the point P.

The part of the integral contributed by the volume element (dv) at M ≡ (X, Y, Z) is—

$$D_M = -\frac{1}{4\pi} \frac{f\left[\left(t - \frac{r}{s}\right), X, Y, Z\right]}{r} dv$$

$$= -\frac{1}{4\pi} \frac{f\left[\left(t - \frac{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}}{s}\right), X, Y, Z\right]}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}} dv \qquad (16)$$

Taking the second derivative of D_M with respect to (t),

$$\frac{d^2}{dt^2} D_M = -\frac{1}{4\pi} \frac{f''\left[\left(t - \frac{r}{s}\right), X, Y, Z\right]}{r} dv \qquad (17)$$

(See the footnote* for the meaning of f″ [(t − r/s),X,Y,Z])

*In the expression $f\left[\left(t - \frac{r}{s}\right), X, Y, Z\right]$, let $\left(t - \frac{r}{s}\right)$ be represented by (u). Then—

$$\frac{d}{dx} f\left[\left(t - \frac{r}{s}\right), X, Y, Z\right] = \frac{du}{dx}\frac{d}{du} f\left[u, X, Y, Z\right]$$

$$= \frac{du}{dx} f'\left[u, X, Y, Z\right]$$

and

$$\frac{d^2}{dx^2} f\left[\left(t - \frac{r}{s}\right), X, Y, Z\right] = \frac{d}{dx}\left\{\frac{du}{dx} f'\left[u, X, Y, Z\right]\right\}$$

$$= \frac{d^2u}{dx^2} f'\left[u, X, Y, Z\right] + \left(\frac{du}{dx}\right)^2 f''\left[u, X, Y, Z\right]$$

$$= \frac{d^2u}{dx^2} f'\left[\left(t - \frac{r}{s}\right), X, Y, Z\right] + \left(\frac{du}{dx}\right)^2 f''\left[\left(t - \frac{r}{s}\right), X, Y, Z\right]$$

Taking the second derivative of D_M with respect to (x),

$$\frac{d^2}{dx^2} D_M = -\frac{dv}{4\pi}\left\{f''\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\frac{(X-x)^2}{s^2r^3}\right.$$
$$+ f'\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\left[\frac{3(X-x)^2}{sr^4} - \frac{1}{sr^2}\right]$$
$$\left.+ f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\left[\frac{3(X-x)^2}{r^5} - \frac{1}{r^3}\right]\right\}$$

In like manner, the following derivatives are obtained:

$$\frac{d^2}{dy^2} D_M = -\frac{dv}{4\pi}\left\{f''\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\frac{(Y-y)^2}{s^2r^3}\right.$$
$$+ f'\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\left[\frac{3(Y-y)^2}{sr^4} - \frac{1}{sr^2}\right]$$
$$\left.+ f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\left[\frac{3(Y-y)^2}{r^5} - \frac{1}{r^3}\right]\right\}$$

and

$$\frac{d^2}{dz^2} D_M = -\frac{dv}{4\pi}\left\{f''\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\frac{(Z-z)^2}{s^2r^3}\right.$$
$$+ f'\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\left[\frac{3(Z-z)^2}{sr^4} - \frac{1}{sr^2}\right]$$
$$\left.+ f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]\left[\frac{3(Z-z)^2}{r^5} - \frac{1}{r^3}\right]\right\}$$

Adding

$$\left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right] D_M = -\frac{1}{4\pi s^2}\,\frac{f''\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]}{r}\,dv \tag{18}$$

Hence from equations (17) and (18),

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_M = 0$$

As the volume element at M represents any volume element save the element surrounding P, it follows that the Dalembertian of all that portion of D which is contributed by volume elements other than the element immediately surrounding P is zero.

There remains to be considered the value contributed to the integral by the volume element N immediately surrounding the point P. Let this element be taken as a small spherical volume of radius R. To establish our proposition, the Dalembertian of the quantity which is contributed to D by this volume element must be demonstrated to equal the value of the f (t, x, y, z) at P.

If the $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ is zero within the sphere N, this portion of space contributes nothing to D. Therefore, if at P the f (t, x, y, z) is zero, the Dalembertian of D for the point P is zero, and the value obtained for D by the summation expressed by equation (15) satisfies the differential relation expressed in equation (14).

16. Proof for Charges at the Point

If the, $f\left[\left(t-\frac{r}{s}\right), X,\dot{Y},Z\right]$ is not zero within the small sphere N surrounding the point P — in other words, if this sphere contains a charge—the value D_N contributed to D by the values of $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ within this spherical volume must be determined, and, as previously stated, to establish our proposition, the Dalembertian of D_N must be shown to equal the value of the f (t, x, y, z) at the point P.

The value of D_N is defined by the equation,

$$D_N = -\frac{1}{4\pi}\int^{\text{Vol. of the sphere}} \frac{f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]}{r}dv$$

In the first place, it is to be noted that the infinite value assumed by the integrand when r equals zero does not mean that the integral D_N is infinite. This may be demonstrated as follows:

Over the space within a sphere of infinitesimal radius R the $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ will, for any given instant of time, have substantially the same values at all points within the sphere. Or, at any rate, the values of the function for the different points within the sphere may be conceived to lie between a maximum value f_1 and a minimum value f_2. By letting the radius of the sphere approach zero, these values may be made to differ by an infinitesimal amount from the value of the function at the center of the sphere, namely f (t, x, y, z).

Consequently, the $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ may be imagined to have the uniform value f (t, x, y, z) at all points within the sphere. The value of D_N may then be computed by dividing

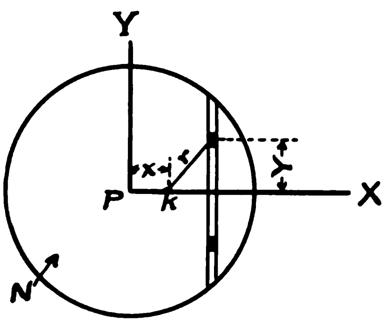

FIG. 10

up the spherical volume into spherical shells of thickness (dr) and carrying out the indicated integration.

Whence

$$D_N = -\frac{1}{4\pi}\int_0^R \frac{f\,(t,x,y,z)\;4\pi\,r^2\,dr}{r}$$

$$= -\frac{R^2}{2}\,f\,(t,x,y,z) \qquad (19)$$

The value of D_N is therefore not only finite, but it can be caused to decrease without limit by decreasing the radius of the sphere without limit.

For the purpose of calculating the Dalembertian of the value D_N which is contributed to the retarded potential by the space within the sphere N, let the origin of the system of coordinates be transferred to the center (P) of the sphere. Then let the retarded potential be calculated—not for the point P—but for a point K, Fig. 10, displaced along the X axis by an infinitesimal distance (x) from the center. (This is for the purpose of obtaining the expression for D_N in such a form that the value of $\frac{d^2 D_N}{dx^2}$ may be calculated.)

Imagine the volume of the sphere N to be made up of plane slices of thickness (dX), and these slices in turn to be constituted of circular rings of radius (Y), as shown in Fig. 10.

The expression for the integral D_N now takes the form:

$$D_N = -\frac{1}{4\pi}\int_{X=-R}^{X=x}\int_{Y=o}^{Y=\sqrt{R^2-X^2}} \frac{f\,(t,x,y,z)\;2\pi\;YdYdX}{\sqrt{(x-X)^2+Y^2}}$$

$$-\frac{1}{4\pi}\int_{X=x}^{X=R}\int_{Y=o}^{Y=\sqrt{R^2-X^2}} \frac{f\,(t,x,y,z)\;2\pi\;YdYdX}{\sqrt{(X-x)^2+Y^2}} \qquad (20)$$

Integrating,

$$D_N = -\left(\frac{R^2}{2}-\frac{x^2}{6}\right) f\,(t,x,y,z) \qquad (21)$$

Taking the second derivative of D_N with respect to x and letting R approach zero, the second derivative approaches the following limit:

$$\frac{d^2D_N}{dx^2} = \frac{1}{3}\,f\,(t,x,y,z)$$

In like manner it may be shown that

$$\frac{d^2D_N}{dy^2} = \frac{1}{3}\,f\,(t,x,y,z)$$

and $$\frac{d^2D_N}{dz^2} = \frac{1}{3}\,f\,(t,x,y,z)$$

Taking the second derivative of (19) or (21) with respect to (t) and letting R approach zero, the second derivative approaches the following limit.

$$\frac{d^2 D_N}{dt^2} = 0$$

Therefore

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_N = f\ (t,x,y,z)$$

This establishes the proposition that the values of D determined by equation (15) satisfy the differential relation expressed in equation (14), or the values of a time and point function D whose Dalembertian is a given time and point function f (t, x, y, z) may be found by the operation of forming the retarded potential of the f (t, x, y, z).

17. Proof that the Forces May be Calculated From the "Retarded Displacement Potential" and the "Retarded Vector Potential."

By equations (10a) and (15),

$$D_1 = -\frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1)\right]_{(t-r/s)}^{*}}{r}\, dv \qquad (22)$$

Splitting the right hand member of (22) into two parts, equation (22) may be written

$$D_1 = D_1' + D_1'' = -\frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}\right]_{(t-r/s)}}{r}\, dv$$

$$-\frac{1}{4\pi}\int \frac{\left[\frac{1}{s^2}\frac{d}{dt}(\rho V_1)\right]_{(t-r/s)}}{r}\, dv \qquad (22a)$$

In which, D_1' represents the 1st, and D_1'' the 2nd term.

* For the meaning of the subscript (t − r/s), see the footnote to equation (26).

In like manner,

$$D_2 = D_2' + D_2'' = -\frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dy}\right]_{(t-r/s)}}{r} dv$$

$$-\frac{1}{4\pi}\int \frac{\left[\frac{1}{s^2}\frac{d}{dt}(\rho V_2)\right]_{(t-r/s)}}{r} dv \qquad (22b)$$

$$D_3 = D_3' + D_3'' = -\frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dz}\right]_{(t-r/s)}}{r} dv$$

$$-\frac{1}{4\pi}\int \frac{\left[\frac{1}{s^2}\frac{d}{dt}(\rho V_3)\right]_{(t-r/s)}}{r} dv \qquad (22c)$$

By equations (13) and (15),

$$H_1 = -\frac{1}{4\pi}\int \frac{-\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right]_{(t-r/s)}}{r} dv \qquad (23a)$$

$$H_2 = -\frac{1}{4\pi}\int \frac{-\left[\frac{d}{dz}(\rho V_1) - \frac{d}{dx}(\rho V_3)\right]_{(t-r/s)}}{r} dv \qquad (23b)$$

$$H_3 = -\frac{1}{4\pi}\int \frac{-\left[\frac{d}{dx}(\rho V_2) - \frac{d}{dy}(\rho V_1)\right]_{(t-r/s)}}{r} dv \qquad (23c)$$

The right hand members of equations (22) and (23) contain terms of the type $\frac{d\rho}{dx}$ and $\frac{d}{dy}(\rho V_3)$. Now the values of these terms depend upon the actual space distribution of the moving charge. Thus far, we have not specified in detail the manner in which the moving charges of Fig. 5 are distributed. All that has been stated of Fig. 5 is that two charges, a positive charge Q and a negative charge -Q, are distributed in some manner over two sets of thin circular rings which move up and down in a simple harmonic manner. The values of D_1, D_2, D_3 and of H_1, H_2, H_3 might be computed by making any arbitrary assumptions as to the distribution of the charges over the rings, subject only to the condition that the total charge is to equal +Q on one set and −Q on the other set of rings. It is far more convenient, however, to follow a procedure which involves no specific assumptions of this kind. This procedure is as follows:

The components of the displacement and of the magnetic force may be derived from two auxiliary point functions defined as follows:

Let the "retarded displacement potential*" Φ be defined as a scalar point function satisfying the differential relation,

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]\Phi = -\rho \tag{24}$$

Also let the "retarded vector potential" $\mathbf{A}$ be defined as a vector point function satisfying the differential relation,

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]\mathbf{A} = -\rho\mathbf{V} \tag{25}$$

In other words, Φ and $\mathbf{A}$ are defined as quantities whose Dalembertians are to be equal to the negative of the volume density and the negative of the current density respectively. Therefore the values of Φ and $\mathbf{A}$ will be found by the operation of forming the retarded potentials of $-\rho$ and $-\rho\mathbf{V}$. Thus

$$\Phi = \frac{1}{4\pi}\int \frac{\left[\rho\right]_{(t-r/s)}}{r}\,dv \dagger \tag{26}$$

$$\mathbf{A} = \frac{1}{4\pi}\int \frac{\left[\rho\mathbf{V}\right]_{(t-r/s)}}{r}\,dv \tag{27}$$

The components of the vector $\mathbf{A}$ will be given by the equations,

$$A_1 = \frac{1}{4\pi}\int \frac{\left[\rho V_1\right]_{(t-r/s)}}{r}\,dv \tag{27a}$$

$$A_2 = \frac{1}{4\pi}\int \frac{\left[\rho V_2\right]_{(t-r/s)}}{r}\,dv \tag{27b}$$

* "The *displacement* potential" should not be confused with "*the* potential." The gradient of the former gives displacement and of the latter, electric force. The value of the former is p (the permittivity) times the latter.

†The subscripts $(t-r/s)$ appearing in equations (26) and (27) indicate that in finding the retarded potential at a point P for the instant of time (t), any element of volume is to be multiplied by the value of ρ or $\rho\mathbf{V}$ in the element at the instant $(t-r/s)$.

$$A_3 = \frac{1}{4\pi}\int \frac{\left[\rho V_3\right]_{(t-r\ s)}}{r}\,dv \qquad (27c)$$

It may be shown that,

$$D_1' = -\frac{d\Phi}{dx} \text{ and } D_1'' = -\frac{1}{s^2}\frac{d}{d.}A_1 \text{ or } D_1 = -\frac{d\Phi}{dx} - \frac{1}{s^2}\frac{d}{dt}A_1 \qquad (28a)$$

$$D_2' = -\frac{d\Phi}{dy} \text{ and } D_2'' = -\frac{1}{s^2}\frac{d}{d!}A_2 \text{ or } D_2 = -\frac{d\Phi}{dy} - \frac{1}{s^2}\frac{d}{dt}A_2 \qquad (28b)$$

$$D_3' = -\frac{d\Phi}{dz} \text{ and } D_3'' = -\frac{1}{s^2}\frac{d}{dt}A_3 \text{ or } D_2 = -\frac{d\Phi}{dz} - \frac{1}{s^2}\frac{d}{dt}A_3 \qquad (23c)$$

Or in vector notation,

$$\mathbf{D} = -\text{ grad }\Phi - \frac{1}{s^2}\frac{d}{dt}\mathbf{A} \qquad (28)$$

And that

$$H_1 = \frac{dA_3}{dy} - \frac{dA_2}{dz} = \text{X component of curl of } \mathbf{A} \qquad (29a)$$

$$H_2 = \frac{dA_1}{dz} - \frac{dA_3}{dx} = \text{Y component of curl of } \mathbf{A} \qquad (29b)$$

$$H_3 = \frac{dA_2}{dx} - \frac{dA_1}{dy} = \text{Z component of curl of } \mathbf{A} \qquad (29c)$$

Or in vector notation,

$$\mathbf{H} = \text{curl } \mathbf{A} \qquad (29)$$

The proof of these statements is as follows:

18. TO SHOW THAT $D_1' = -\frac{d\Phi}{dx}$

Let P, Fig. 11, represent any point in space and K a second point displaced from P in a direction parallel to the X axis by the infinitesimal amount (dx). Let the values of the retarded potential Φ at these two points for a given instant of time be represented by Φ_k and Φ_p.

Let the operation of forming the retarded potentials at the points P and K for the instant (t) be visualized as carried out in the following manner:

First: Visualize all space as divided into volume elements, with radii extending from the point P to all those volume elements which contain a charge at the instant $(t-r/s)$. A few of these volume elements and radii have been shown in Fig. 11. The value of Φ_p is the result of carrying out the summation expressed by equation (26) over this system.

Second: Now visualize a second set of radii (indicated by the dotted lines in Fig. 11) all issuing from the point K and

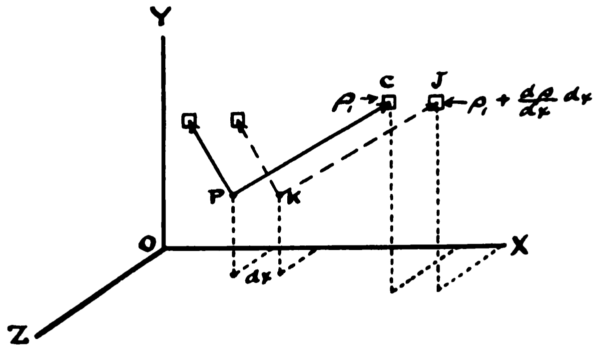

Fig. 11

drawn parallel and equal to the radii issuing from point P. The value Φ_k is the result of carrying out the summation expressed by equation (26) over this system.

How do the summations for Φ_p and Φ_k differ? Every radius with its terminal volume element in one system can be matched by a corresponding radius and volume element in the other. The only difference is in the density of the charge ρ in corresponding volume elements. If the volume density in any element belonging to the point P system (as the element C of Fig. 11) is ρ_1, the volume density in the corresponding element of the point K system (J of Fig. 11) is $\left(\rho_1 + \frac{d\rho}{dx} dx\right)$. Therefore the values of Φ_p and Φ_k will be given by summations involving identical combinations of (r) and (dv) associated with the volume density ρ in the P system and with $\left(\rho + \frac{d\rho}{dx} dx\right)$ in the K system.

$$\Phi_K = \frac{1}{4\pi}\int \frac{\left[\rho + \frac{d\rho}{dx}dx\right]_{(t-r/s)}}{r} dv$$

$$= \frac{1}{4\pi}\int \frac{\left[\rho\right]_{(t-r/s)}}{r} dv + \frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}dx\right]_{(t-r/s)}}{r} dv$$

and

$$\Phi_P = \frac{1}{4\pi}\int \frac{\left[\rho\right]_{(t-r/s)}}{r} dv$$

Therefore

$$\Phi_K - \Phi_P = \frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}dx\right]_{(t-r/s)}}{r} dv$$

Or $$\frac{\Phi_k - \Phi_p}{dx} = \frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}\right]_{(t-r/s)}}{r} dv$$

But $\frac{\Phi_k - \Phi_p}{dx}$ is $\frac{d\Phi}{dx}$

and from equation (22a)

$$D_1' = -\frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}\right]_{(t-r/s)}}{r} dv$$

Therefore

$$D_1' = -\frac{d\Phi}{dx}$$

In like manner it may be shown that

$$D_2' = -\frac{d\Phi}{dy}$$

and $D_3' = -\frac{d\Phi}{dz}$

19. To Show That $D_1'' = -\frac{1}{s^2}\frac{d}{dt}A_1$

From equation (27a), the X component A_1 of the retarded vector potential is

$$A_1 = \frac{1}{4\pi}\int \frac{\left[\rho V_1\right]_{(t-r/s)}}{r} dv$$

By visualizing the operations involved in finding the values of A_1 at any point P for two instants of time t_1 and $(t_1 + dt)$, it may be seen that

$$\frac{d}{dt}A_1 \text{ or } \frac{d}{dt}\left[\frac{1}{4\pi}\int \frac{\left[\rho V_1\right]_{(t-r/s)}}{r} dv\right]$$

$$= \frac{1}{4\pi}\int \frac{\left[\frac{d}{dt}\left(\rho V_1\right)\right]_{(t-r/s)}}{r} dv$$

But from equation (22a),

$$D_1'' = -\frac{1}{4\pi}\int \frac{\left[\frac{1}{s^2}\frac{d}{dt}\left(\rho V_1\right)\right]_{(t-r/s)}}{r} dv$$

Therefore

$$D_1'' = -\frac{1}{s^2}\frac{d}{dt}A_1$$

In like manner it may be seen that

$$D_2'' = -\frac{1}{s^2}\frac{d}{dt}A_2$$

$$\text{and } D_3'' = -\frac{1}{s^2}\frac{d}{dt}A_3$$

20. To Show That $H_1 = \frac{dA_3}{dy} - \frac{dA_2}{dz}$

By equation (27c),

$$A_3 = \frac{1}{4\pi}\int \frac{\left[\rho V_3\right]_{(t-r/s)}}{r} dv$$

By visualizing the summations involved in finding the values of A_3 at a point P and at a second point K displaced from P in a direction parallel to the Y axis by the infinitesimal amount (dy), it may be seen that

$$\frac{d}{dy}A_3 = \frac{1}{4\pi}\int \frac{\left[\frac{d}{dy}(\rho V_3)\right]_{(t-r/s)}}{r} dv$$

In a similar way it may be seen that

$$\frac{d}{dz}A_2 = \frac{1}{4\pi}\int \frac{\left[\frac{d}{dz}(\rho V_2)\right]_{(t-r/s)}}{r} dv$$

But from equation (23a),

$$H_1 = \frac{1}{4\pi}\int \frac{\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right]_{(t-r/s)}}{r} dv$$

Therefore

$$H_1 = \frac{dA_3}{dy} - \frac{dA_2}{dz} \qquad (29a)$$

Similar demonstrations may be used to establish equations (29b) and (29c).

21. To Summarize:

If a system of charges moves in space in a known manner, the electric displacement **D** and the magnetic force **H** at any point in space may be derived from the retarded displacement potential Φ and the retarded vector potential **A** by the equations,

$$\mathbf{D} = -\text{grad}\,\Phi - \frac{1}{s^2}\frac{d}{dt}\mathbf{A} \qquad (29)$$

$$\mathbf{H} = \text{curl}\,\mathbf{A} \qquad (29)$$

The values of the retarded potentials Φ and **A** are to be computed by the equations,

$$\Phi = \frac{1}{4\pi}\int \frac{[\rho]_{(t-r/s)}}{r} dv \qquad (26)$$

$$\mathbf{A} = \frac{1}{4\pi}\int \frac{[\rho \mathbf{V}]_{(t-r/s)}}{r} dv \qquad (27)$$

IV. EXPRESSIONS FOR THE FORCES AT GREAT DISTANCES FROM THE RADIATOR

22. The Field at Points Near the Earth's Surface

We proceed to compute the electric displacement **D** and the magnetic force **H** at a point P near the earth's surface and at a great distance (100 wave lengths or more) from the radiator shown in Fig. 5. * In order that we may have specific conditions to discuss, let the radiator be assumed to be of the following proportions:

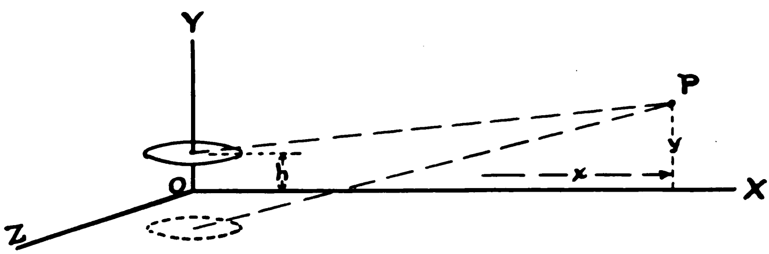

Fig. 12

Table I

Radius of charged areas	165 meters
Maximum elevation attained by the charged area	60 meters
Frequency	80,000 cycles
Distance to the point P	600 kilometers
Elevation of the point P above SS	90 meters
Maximum potential of + area to earth	100 kv.

The radiating system has been redrawn in Fig. 12. In this figure, the XZ plane has been taken to coincide with the neutral surface SS of Fig. 5; the XY plane has been passed through the point P ≡ (x, y, O); and the positive and negative charges on the two sets of rings are assumed to be

*In the subsequent discussion, the following approximations are used: (a) the earth's surface is treated—not as a spherical surface—but as a plane surface, (b) the resistance losses in the surface of the earth and the absorption losses in the atmosphere have been neglected.

symmetrically distributed about the Y axis. The two sets of rings carry the charges +Q and −Q coulombs, and their respective elevations above the XZ plane are expressed by the equations:

$$h_p = h_o \cos \omega t \tag{30a}$$

$$h_n = -h_o \cos \omega t \tag{30b}$$

23. The Retarded Displacement Potential Φ

As the element of charge (dQ) located at the center of the positive plate moves up and down in the simple harmonic manner expressed by equation (30a), its distance (r) from the point P varies in the manner expressed by the following equation.

$$r = \left[x^2 + (h_o \cos \omega t - y)^2\right]^{\frac{1}{2}}$$

$$= x\left[1 + \frac{(h_o \cos \omega t - y)^2}{2x^2} + \ldots\ldots\right]$$

Let us first draw the curves showing the steady state values of the displacement potential at P which correspond to the position of the charges at any instant of time, and then proceed to consider in what respect these curves must be altered to make them represent the retarded potential. By the "steady state value of the potential at P corresponding to the position of the charges at any instant" is meant the potential which would be assumed by the point P if the charges ever after remain fixed in the position they have at the instant under consideration.

The value at point P of the steady state displacement potential due to the various positions of the charge (dQ) is given by the equation,

$$d\Phi_p = \frac{1}{4\pi} \frac{dQ}{x\left[1 + \frac{(h_o \cos \omega t - y)^2}{2x^2}\right]}$$

$$= \frac{dQ}{4\pi x}\left[1 - \frac{(h_o \cos \omega t - y)^2}{2x^2}\right]$$

$$= \frac{dQ}{4\pi x}\left[1 - \frac{1}{2x^2}(h_o^2 \cos^2 \omega t - 2h_o y \cos \omega t + y^2)\right] \tag{31}$$

As the element of charge $(-dQ)$ located at the center of the negative plate moves up and down, its distance (r) from the point P varies in the manner expressed by the following equation,

$$r = \left[x^2 + (-h_o \cos \omega t - y)^2\right]^{1/2}$$

The value at the point P of the steady state displacement potential due to the various positions of this charge, is given by the equation,

$$d\,\Phi_n = -\frac{dQ}{4\pi x}\left[1 - \frac{1}{2x^2}(h_o^2 \cos^2 \omega t + 2h_o y \cos \omega t + y^2)\right] \quad (32)$$

The resultant displacement potential at P due to the corresponding elements of charge at the centers of the positive and negative sets of rings is

$$d\,\Phi = d\,\Phi_p + d\,\Phi_n = \frac{(dQ)\ h_o y}{2\pi x^3} \cos \omega t \quad (33)$$

Curves showing the variation in time of the values of the terms of equations (31), (32), and (33) have been drawn (but not to scale) in Figures 13, 14, and 15 respectively. For example, the right hand member of equation (31) consists of the constant term $\frac{dQ}{4\pi x}\left(1 - \frac{y^2}{2x^2}\right)$ upon which two very small variable terms are superimposed; the variable term $\frac{dQ}{4\pi x}\,\frac{h_o y \cos \omega t}{x^2}$ is of the same frequency as the frequency of vibration of the moving charge, and the term $-\frac{dQ}{4\pi x}$ $\frac{h^2_o \cos^2 \omega t}{2x^2}$ is of double this frequency. These terms are represented by curves A, B, and C of Fig. 13.

This question now arises: How must these curves be modified in order to make them represent the values of the retarded displacement potential at P?

At the instant at which the charges pass in opposite directions across the neutral surface (as the instant t_1), the distance of the elementary charges from P is $\sqrt{x^2 + y^2}$. Now the

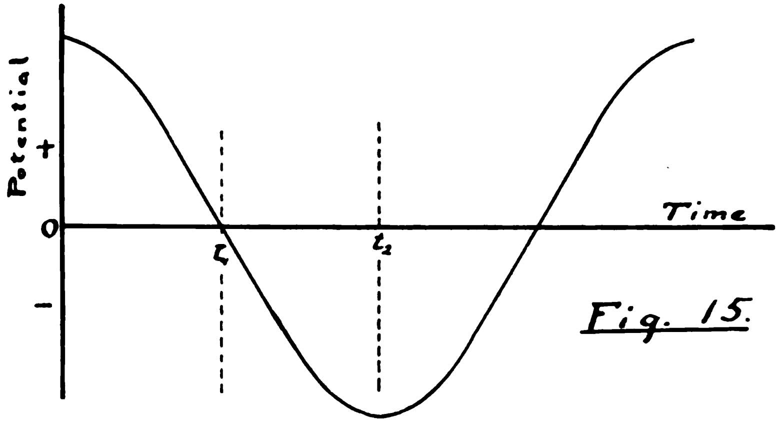

Fig. 15.

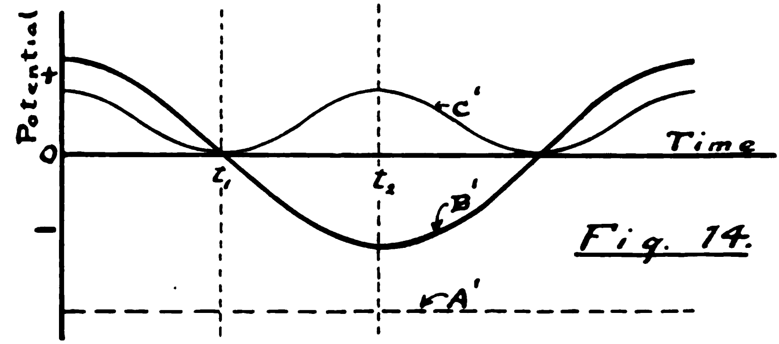

Fig. 14.

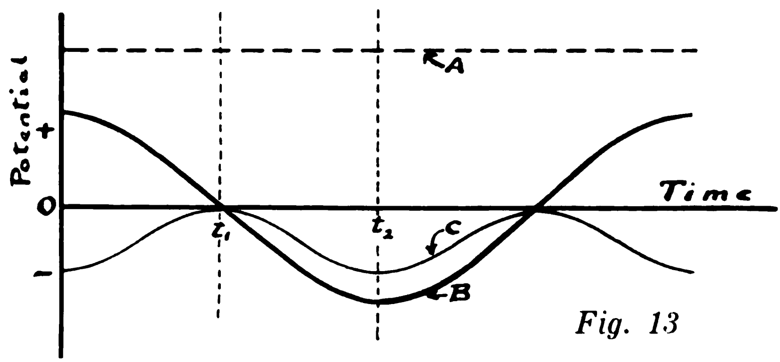

Fig. 13

effect corresponding to the location of any element of the charge at the instant (t_1) is felt at P (r/s) seconds later. Therefore it follows that the potential shown by Figures 13, 14, and 15 for the instant (t_1) represents the retarded potential at P at the instant $\left(t_1 + \frac{\sqrt{x^2+y^2}}{s}\right)$.

During the quarter cycle from (t_1) to (t_2), the positive charge moves farther and farther away from P and the negative charge approaches P. At the instant (t_2), the positive charge is at its maximum distance and the negative charge is at its minimum distance from P. At this instant the positive charge is approximately 1.2 cm. farther from, and the negative charge is approximately .6 cm. nearer to P than they were at the instant (t_1).

Therefore, the potential shown in Fig. 13 for the instant (t_2) represents the retarded potential at P at the instant $\left[t_2 + \frac{\sqrt{x^2+y^2}}{s} + \frac{1.2}{s}\right]$, while the potential shown in Fig. 14 for the instant (t_2) represents the retarded potential at P at the instant $\left[t_2 + \frac{\sqrt{x^2+y^2}}{s} - \frac{.6}{s}\right]$. That is to say, in altering the curves in Figs. 13, 14, and 15 in order to make them represent the retarded potentials, the potentials shown for the instant (t_1) in Figs. 13 and 14 must be shifted back in time by $\frac{\sqrt{x^2+y^2}}{s}$ seconds, the potential shown for the instant (t_2) in Fig. 13 must be shifted $\frac{\sqrt{x^2+y^2}}{s} + \frac{1.2}{s}$ seconds, and the potential shown for the instant (t_2) in Fig. 14 must be shifted $\frac{\sqrt{x^2+y^2}}{s} - \frac{.6}{s}$ seconds.

There is, therefore, between points on the Fig. 13 and Fig. 14 curves a relative shift which never exceeds $\left(\frac{1.2}{s} + \frac{.6}{s}\right)$ seconds, or $6\ 10^{-11}$ seconds. Now a shift of $6\ 10^{-11}$ seconds in time corresponds at a frequency of 80,000 cycles to a shift of only five one-millionths of a cycle or .0018 degrees.

From this it follows that for all practical purposes,* the curves of Figures 13, 14, and 15 will represent the retarded potentials at P if they are shifted in time by $\frac{\sqrt{x^2+y^2}}{s}$ seconds, or substantially by $\frac{x}{s}$ seconds. From equation (33), it follows that the value of the retarded displacement potential at P due to the corresponding elements of charge at the centers of the positive and negative sets of rings is given by the expression:

$$d\Phi_r = \frac{(dQ)\, h_0 y}{2\pi x^3} \cos \omega \left(t - \frac{x}{s}\right) \tag{34}$$

In like manner, every element of the positive charge may be paired with its corresponding element of the negative charge; any pair may be seen to contribute to the value of the retarded potential at P the quantity

$$d\Phi = \frac{(dQ)\, h_0 y}{2\pi\, (x-X)^3} \cos \omega \left(t - \frac{(x-X)}{s}\right) \tag{35}$$

in which, X is the abscissa of the pair of charges under consideration. Thus, the resultant potential at P will be the sum of a great number of harmonic functions of the same frequency but differing somewhat in phase. The maximum difference in phase will be equal to the interval required for the transmission of the disturbance over a distance equal to the diameter of the radiator. This interval equals $330 \div 3\ 10^8$ or $1.1\ 10^{-6}$ seconds, which at a frequency of 80,000 cycles means a phase difference of .088 of a cycle, or 32 degrees.

Since the charges on the radiator have been assumed to be symmetrically disposed about the axis, and since the contribution made by any pair of elements to the potential

*It should be borne in mind that none of these arguments or conclusions apply if the point P is within a few wave lengths of the radiator, or if the line OP from the radiator to P makes an angle greater than one or two degrees with a horizontal plane. If OP makes an appreciable angle with the horizontal, these simple considerations require modification. See Appendix A for the forces at elevated points.

does not differ in phase by more than 16 degrees from the contribution of the central pair of charges, it follows that the value of the retarded displacement potential at P will, to within a few tenths of one per cent, be given by the expression:

$$\Phi = \frac{Q\, h_o\, y}{2\pi x^3} \cos \omega \left(t - \frac{x}{s}\right) \tag{36}$$

24. THE RETARDED VECTOR POTENTIAL $\mathbf{A}$

The retarded vector potential $\mathbf{A}$ is defined by the equation,

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\left[\rho \mathbf{V}\right]_{(t-r/s)}}{r}\, dv \tag{27}$$

Now, in Fig. 5, the charges move only in a direction parallel to the Y axis. Therefore, the X and Z components of $\mathbf{V}$ are zero, or the vector potential $\mathbf{A}$ is a vector parallel to the Y axis. For the positive charge, the Y component of the velocity $\mathbf{V}$ is,

$$V_{2p} = \frac{dh}{dt} = -h_o \omega \sin \omega t$$

and for the negative charge, the Y component of the velocity is,

$$V_{2n} = h_o \omega \sin \omega t$$

Since the moving charges are symmetrically disposed about the Y axis, and since the vector potential contributed by any element does not differ in phase by more than 16 degrees from the potential contributed by the charges moving along the Y axis, it follows that the value of the retarded vector potential at P will, to within a few tenths of one per-cent, be given by the expression

$$\mathbf{A} = A_2 = \frac{Q}{4\pi x}\left\{-h_o\omega \sin \omega \left(t - \frac{x}{s}\right)\right\} + \frac{-Q}{4\pi x}\left\{h_o\omega \sin \omega \left(t - \frac{x}{s}\right)\right\}$$

$$\text{or } A_2 = -\frac{Q\, h_o \omega}{2\pi x} \sin \omega \left(t - \frac{x}{s}\right) \tag{37}$$

25. THE DISPLACEMENT AND POTENTIAL GRADIENT AT P. From equation (28), the displacement is given by,

$$\mathbf{D} = -\text{grad}\,\Phi - \frac{1}{s^2}\frac{d}{dt}\mathbf{A} \qquad (28)$$

Taking the derivatives of Φ, as expressed in equation (36),

$$\frac{d\Phi}{dx} = \frac{Qh_o}{2\pi}\left[-\frac{3y}{x^4}\cos\omega\left(t-\frac{x}{s}\right)+\frac{y}{x^3}\frac{\omega}{s}\sin\omega\left(t-\frac{x}{s}\right)\right]$$

$$\frac{d\Phi}{dy} = \frac{Qh_o}{2\pi x^3}\cos\omega\left(t-\frac{x}{s}\right)$$

$$\frac{d\Phi}{dz} = 0$$

At points of the neutral surface (the XY plane), $\frac{d\Phi}{dx}$ is zero since (y) is zero. At points near the XY plane, $\frac{d\Phi}{dx}$ is negligibly small in comparison with $\frac{d\Phi}{dy}$. That is to say, the Y component of the gradient of Φ or D_2' is the only component of appreciable magnitude at points near the neutral surface. The Y component of the gradient of Φ is given by the expression,

$$D_2' = -\frac{d\Phi}{dy} = -\frac{Qh_o}{2\pi x^3}\cos\omega\left(t-\frac{x}{s}\right)$$

Since the X and Z components of the vector potential are zero, and by equation (37),

$$A_2 = -\frac{Qh_o\omega}{2\pi x}\sin\omega\left(t-\frac{x}{s}\right) \qquad (37)$$

Therefore

$$D_1'' = D_3'' = 0$$

and

$$D_2'' \text{ or } -\frac{1}{s^2}\frac{d}{dt}A_2 = \frac{Qh_o\omega^2}{2\pi x s^2}\cos\omega\left(t-\frac{x}{s}\right)$$

Therefore the displacement at point P is parallel to the Y axis, and its value (vertically upward) is given by the expression,

$$D_2 = D_2' + D_2'' = \frac{Q\,h_o}{2\pi}\left[\frac{\omega^2}{s^2 x} - \frac{1}{x^3}\right]\cos\omega\left(t - \frac{x}{s}\right) \quad (38)$$

The potential gradient at P is likewise parallel to the Y axis and its value (vertically upward) is,

$$F_2 = \frac{D_2}{p} = \frac{Q\,h_o}{2\pi p}\left[\frac{\omega^2}{s^2 x} - \frac{1}{x^3}\right]\cos\omega\left(t - \frac{x}{s}\right) \quad (39)$$

26. THE MAGNETIC FORCE AT P

From equation (29), the magnetic force $\mathbf{H}$ is given by,

$$\mathbf{H} = \text{curl}\ \mathbf{A} \quad (29)$$

Now, at all points in space the X and Z components of the vector $\mathbf{A}$ are zero, and at the point P, the value of $\mathbf{A}$ is,

$$\mathbf{A} = A_2 = -\frac{Q\,h_o\omega}{2\pi x}\sin\omega\left(t - \frac{x}{s}\right) \quad (37)$$

Therefore,

$$H_1 \text{ or } \frac{d\,A_3}{dy} - \frac{d\,A_2}{dz} = 0$$

$$H_2 \text{ or } \frac{d\,A_1}{dz} - \frac{d\,A_3}{dx} = 0$$

$$H_3 \text{ or } \frac{d\,A_2}{dx} - \frac{d\,A_1}{dy} \text{ is,}$$

$$H_3 = \frac{Q\,h_o}{2\pi}\left[\frac{\omega^2}{x\,s}\cos\omega\left(t - \frac{x}{s}\right) + \frac{\omega}{x^2}\sin\omega\left(t - \frac{x}{s}\right)\right] \quad (40)$$

That is to say, the magnetic force is exerted in a direction which is normal to the plane determined by the point P and the axis of the radiator. The magnetic force is exerted along circular paths which are centered upon the axis of the radiator, and which lie in planes normal to this axis.

The magnetic flux density at P will be given by the expression:

$$B_3 = \frac{Q\,h_o\mu}{2\pi}\left[\frac{\omega^2}{xs}\cos\omega\left(t - \frac{x}{s}\right) + \frac{\omega}{x^2}\sin\omega\left(t - \frac{x}{s}\right)\right] \quad (41)$$

V. APPLICATION OF THE EXPRESSIONS FOR THE FORCES AT DISTANT POINTS

27. The Electric Force

The expression for the electric force at P—equation (39)—contains two terms. As the distance from the radiator to the point P is increased, the first term decreases as the first power of the distance and the second term decreases as the cube of the distance. At zero frequency, that is, with the charges stationary, the first term disappears from the equation, and the second term is seen to be identical with the expression previously derived for the gradient in the electrostatic field by the method of images. Hence, we may call the first term the "radiant term" and the second term the "steady-state term."

The radiant and steady-state terms are of equal magnitude at the point $x = \frac{s}{\omega}$, or at a point .159 of a wave length from the radiator. (It should be noted, however, that equations (39) and (40) do not apply to points this close to the radiator). For all points at a greater distance than this from the radiator, the radiant term is the larger term; at great distances, the steady-state term becomes negligibly small. For example: at a point P which is at a distance of 600 kilometers from the radiator specified in Table I, the electric force or potential gradient is

$$F_2 \text{ (in volts per cm.)} = \left[\frac{6.2}{10^5} - \frac{6.2}{10^{11}}\right] \cos 500{,}000\,(t - .002)$$

The magnitude of the radiant term at this point is seen to be one million times as great as the steady-state term.

For points at a great distance from a radiator vibrating at the usual wireless frequencies, the steady-state term may be dropped and equation (39) may be written thus:

$$F_2 = \frac{240\pi^2}{s} \frac{Q\, h_0\, f^2}{x} \cos \omega \left(t - \frac{x}{s}\right) \qquad (42)$$

28. The Magnetic Force

The expression for the magnetic force at P—equation (40)—also consists of two terms. As the distance from the radiator to the point P is increased, the first term decreases as the first power of the distance, and the second term decreases as the square of the distance. Both terms vanish for zero frequency. The coefficient of the second term, namely $\frac{Q\, h_o\, \omega}{2\pi x^2}$ is the magnetic force which would be set up at P if the two charges, instead of oscillating, were to move continuously away from each other with a uniform velocity equal to the velocity they have when crossing the neutral surface. Therefore, we may call the first term the "radiant" term and the second the "steady-current" term.

As in the case of the electric force, the two terms in the magnetic force are of equal magnitude at a point .159 of a wave length from the radiator. At greater distances the radiant term is the larger term. For example: At a point P which is 600 kilometers from the radiator specified in Table I, the magnetic force is

$$H_3 \text{ (in ampere-turns per cm.)} = \frac{1.66}{10^7} \cos \omega \left(t - \frac{x}{s}\right) + \frac{1.66}{10^{10}} \sin \omega \left(t - \frac{x}{s}\right)$$

The magnitude of the radiant term at this point is one thousand times as great as the "steady-current" term. For points at great distances from a radiator vibrating at the usual wireless frequencies the steady current term may be dropped and equation (40) may be written,

$$H_3 = \frac{2\pi}{s} \frac{Q\, h_o\, f^2}{x} \cos \omega \left(t - \frac{x}{s}\right) \qquad (43)$$

29. The Radiant Vector

By Poynting's Theorem, the rate $\mathbf{P}_1$ at which energy streams across unit area at P is the vector product of the electric and magnetic forces at P.

$$\mathbf{P}_1 = \mathbf{F} \times \mathbf{H} \text{ (watts per sq. cm.)}$$

Using the approximate values for F and H given in equations (42) and (43) and taking the vector product, it is seen that the radiant vector P_1 at P points radially away from the oscillator and has the value,

$$P_1 = \frac{480\pi^3}{s^2}\frac{Q^2 h_o^2 f^4}{x^2}\cos^2\omega\left(t-\frac{x}{s}\right) \quad (44)$$

For the specific conditions of Table I, this reduces to

$$P_1 \text{ (in watts per sq. cm.)} = \frac{1.01}{10^{11}}\cos^2\omega\left(t-\frac{x}{s}\right)$$

This result may be arrived at in another way. The electro-potential energy per unit volume at P is

$$\frac{1}{2}\,pF^2 \text{ or } \frac{240\,\pi^3}{s^3}\frac{Q^2 h_o^2 f^4}{x^2}\cos^2\omega\left(t-\frac{x}{s}\right)$$

The electro-kinetic energy per unit volume at P is

$$\frac{1}{2}\,\mu H^2 \text{ or } \frac{240\pi^3}{s^3}\frac{Q^2 h_o^2 f^4}{x^2}\cos^2\omega\left(t-\frac{x}{s}\right)$$

The energy in the electro-potential form per unit volume is seen to equal the energy in the electro-kinetic form, and the total energy per unit volume is $\frac{480\pi^3}{s^3}\frac{Q^2 h_o^2 f^4}{x^2}\cos^2\omega\left(t-\frac{x}{s}\right)$ Since the state of the medium is propagated outward with the velocity (s), the rate at which energy streams across unit area at P will be (s) times the energy per unit volume.

$$s\left[\frac{480\pi^3}{s^3}\frac{Q^2 h_o^2 f^4}{x^2}\cos^2\omega\left(t-\frac{x}{s}\right)\right] = \frac{480\pi^3}{s^2}\frac{Q^2 h_o^2 f^4}{x^2}\cos^2\omega\left(t-\frac{x}{s}\right)$$

This product is seen to be identical with the expression in equation (44).

30. The "Radiation Figure of Merit" of an Antenna

We are now in a position to answer the question raised at the beginning of this discussion, namely: How is the magnitude of the disturbance which is set up in the medium at a distant point P affected by the height of the capacity area of the radiating station?

The expressions for the electric force, for the magnetic force, and for the rate at which energy streams past P—equations (38) to (44)—all contain the factor (Qh_o). That is, the magnitudes of the forces and fluxes are directly proportional to the value of the product of the charge (Q) on the capacity area times the maximum height (h_o) of the capacity area, and the rate at which energy streams past P is proportional to the square of this product.

Now the charge Q on any capacity area may be written as equal to the capacity (C) of the area to earth times the voltage (E) from the charged area to earth, both C and E being the values for the instant when the area is at its point of maximum elevation. Accordingly, (Qh_o) may be replaced by its equivalent (Ch_oE). Now the maximum voltage (E) which may be applied between the capacity area and earth is limited by such considerations as the failure of the insulators and the ionization losses around the wires. This voltage is (within limits) substantially independent of the capacity (C) and elevation (h_o). Possibly the limiting voltage would be somewhat lower for a radiator having the capacity area at a great elevation (150 meters) than for a radiator having the capacity area at a moderate elevation (10 meters), because of the greater mechanical difficulties encountered in insulating an extended area at a great elevation.

From this it follows that if two different antennae of extended area are operated at the same voltage and frequency, then the magnitudes of the electric forces—or of the magnetic forces—at the same distance from the two antennae are proportional to the values of the (Ch) products of the two antennae. The product Ch—the product of the capacity of the extended area times its elevation—may, therefore, be called the *Radiation Figure of Merit* of the antenna.

The question now is, How is the figure of merit of an antenna having an extended capacity area affected by the height of the capacity area above the ground? Imagine the area to be in the form of a circular sheet of radius R, as in Fig. 1. If the radius (R) is large as compared with the elevation (h), the capacity (C) is given approximately by the expression

$$C = \frac{p\pi R^2}{h}$$

$$\text{Whence } Ch = p\pi R^2$$

That is to say, the radiation figure of merit of the antenna is independent of h, or the values of the electric and magnetic forces at distant points are independent of the height of the capacity area. On the other hand, if the radius R is small in comparison with h, the capacity is given approximately by the expression,

$$C = 8pR$$

$$\text{Whence } Ch = 8pRh$$

That is, the radiation figure of merit of the antenna is directly proportional to the elevation h. Between these extreme conditions, the figure of merit of an antenna of given area increases with the elevation of the capacity area, but at a lower rate than the first power of the elevation.

This leads us to a brief consideration of the feasibility of mounting the capacity areas in wireless telegraph stations at a low elevation.

31. A Comparison of Specific Examples of High and Low Antennae

As typical examples of low and high antenna, let the radiator specified in Table I (mounted, however, at only 10 meters elevation) be compared with the antenna of the Government Wireless Station at Arlington, D. C.*

As previously stated, the capacity area of the Arlington Station consists of three approximately horizontal wire harps suspended at an elevation of approximately 150 meters. The capacity of this antenna as determined by measurement is reported to be .0094 microfarads. The antenna cannot, however, be regarded as the equivalent of a radiating area at an elevation of 150 meters and having a capacity of .0094 microfarads. To find the real capacity of an

*For a description of the Arlington Station see D. W. Todd, *Jour. Am. Soc. Naval Engineers*, Vol. 25, February 1913. Also Kintner, Forbes, Kroger and Hogan, *United States Navy Wireless Station* in *Electrical World*, Vol. 61, page 721, April 1913.

area which, mounted at an elevation of 150 meters, would be the radiating equivalent of the Arlington antenna, one must subtract from the measured value of the Arlington antenna to allow for such effects as the following:

1st. A part of the capacity of .0094 microfarads is through the insulators to the steel towers. Any displacement current through the insulators is accompanied by a conduction current flowing up and down the steel supporting towers. Again, a portion of the electrostatic flux from the wire harps terminates on the upper portions of the steel towers; this leads to conduction currents in the towers. These charges running up and down the steel towers move in the opposite direction to the charges in the tail of the antenna and so partially neutralize the radiation from the antenna.

2d. The .0094 microfarads includes the capacity due to displacement from the lower portions of the antenna tail to ground. This capacity area is of course not very effective because of its low elevation.

I should estimate that the radiating power of the Arlington antenna is not greater than that of a capacity area at an elevation of 150 meters and having a capacity of .005 microfarads. The radiation figure of merit of the Arlington antenna is therefore estimated to be .005 x 150 = .75 microfarad-meters. The capacity area specified in Table I is a circular sheet of 165 meters radius mounted at an elevation of 60 meters. The capacity of this sheet to earth is approximately .0125 microfarads, and its figure of merit is .75 microfarad-meters. The radiator of Table I if operated at the same voltage and frequency as the Arlington antenna may, therefore, be expected to set up at distant points a field of the same intensity as the Arlington antenna. Moreover, this sheet of 165 meters radius if mounted at a height of 5 or 10 meters will set up the same field as if mounted at the height of 60 meters.

By mounting the capacity area at an elevation of 10 meters, its capacity is increased to .075 microfarads. This is only 40 per cent less than the capacity of the compressed air condensers used in the primary oscillation circuit of the Arlington station. Now the antenna can very readily be insulated for operating voltages far in excess of the voltages usually applied to the condensers in the primary circuit (30 to 50 Kv). Therefore, if the capacity area is mounted at the moderate elevation of 5 or 10 meters, it is possible to store in the antenna circuit more energy than is generally stored in the condensers of the primary circuits of high power wireless stations. This makes it feasible to dispense with the coupled circuits which are at present used in spark systems of wireless telegraphy and to obtain slightly damped oscillations from a simple oscillating circuit comprising an extended capacity area M, an inductance L, and a spark gap S as shown in Fig. 16.

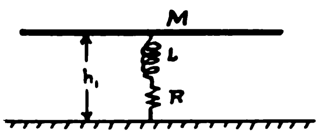

FIG. 17

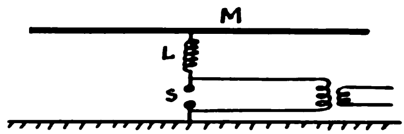

FIG. 16

32. CONSTANTS OF A SENDING STATION

The constants of an antenna of the dimensions given in Table I, save that the capacity area is mounted at an elevation of only ten meters, would be as follows:

TABLE II

CONSTANTS OF AN ANTENNA 10 METERS IN HEIGHT

Radius of capacity area	165	meters
Height of capacity area	10	meters
Capacity of area to earth	.075	microfarads
Assumed operating conditions		
Voltage at moment of discharge	100	peak Kv
Frequency of oscillation	80,000	cycles per sec.
For the above voltage & frequency		
Energy stored	375	joules
Power input at 1000 sparks per second	375	Kw.
Peak value of antenna current	3800	amperes
Critical resistance of oscillatory circuit	53	ohms
Initial rate of radiation	80.	Kw
Radiation resistance (See Appendix B)	.011	ohms
Logarithmic decrement per cycle due to radiation of energy	.0013	
Linear decrement per cycle due to radiation of energy	.13	per cent

It will be noted that the decrement due to the radiation of energy is extremely low; the decrease in the voltage or current due to radiation losses is only .13 of one per cent per cycle. In addition to the radiation loss, the following losses require consideration: the I^2R loss in the conductors, the ionization losses in the air around the conductors, and the losses in the spark gap. The resistance of the conductors to the 80,000 cycle currents can readily be made less than .01 ohms, and the ionization losses made negligibly small. The equivalent resistance of the spark will be of the order of .1 ohms (between .05 and .15 ohms). If the equivalent resistance of the spark is estimated to be .1 of an ohm, the total damping resistance is .12 ohms. This resistance will cause a decrement of only 1.4 per cent per cycle.

The current and the rate of radiation in Table II are calculated for a peak voltage of 100 Kv. This is considerably in excess of the voltages which have hitherto been used with rotary spark gaps. There may be some uncertainty as to the performance of a rotary spark gap at this voltage.

VI. THE LOW ANTENNA FOR RECEIVING PURPOSES

33. The Induced Voltage

Thus far the low antenna has been discussed only as a radiator of electromagnetic waves. This question now arises: Is the effectiveness of an extended antenna as an absorber or receiver of electromagnetic energy independent (within the limits previously stated) of its height above the ground?

Imagine the receiving station to comprise an extended capacity area M, (Fig. 17) mounted at the height (h_1) above the ground, an inductance L, and receiving devices of ohmic resistance R. (The receiving devices may be inductively coupled with the simple series circuit shown in Fig. 17; in this case, R represents the resistance of the receiving devices reduced to the primary circuit). Assume the circuit shown in Fig. 17 to be resonant to the frequency of the sending station, and to be located at the distance (x) from the sending station.

From equation (42), the potential gradient at the receiving station is

$$F_2 = \frac{240\pi^2}{s} \frac{Q\,h_o\,f^2}{x} \cos \omega \left(t - \frac{x}{s}\right) \qquad (42)$$

Therefore the voltage E_1 between the plate M of the receiving station and earth is

$$E_1 = \frac{240\pi^2}{s} \frac{Q\,f^2\,h_o h_1}{x} \cos \omega \left(t - \frac{x}{s}\right) \qquad (45)$$

34. THE BUILDING UP OF THE OSCILLATION

Neglecting minor terms, the equation for the start of an alternating current in a highly oscillatory circuit *resonant* to the impressed frequency, and comprising resistance, inductance, and capacity in series is as follows:

Measuring time from the instant at which the electromotive force is impressed in the circuit, and representing the impressed electromotive force (e) by the equation

$$e = E \cos \omega \,(t - t_1)$$

the equation for the current (i) in the circuit is

$$i = \frac{E}{R}\left(1 - \varepsilon^{-\frac{Rt}{2L}}\right) \cos \omega \,(t - t_1) \qquad (46)$$

in which

R is the resistance of the resonant circuit, and
L is the inductance of the resonant circuit.

Now the resistance of the receiving station consists of—
(1) the useful resistance R_1, that is, the equivalent resistance of the devices which consume a part of the received energy and are actuated thereby.
(2) the unavoidable ohmic resistance R_2 of the antenna conductors and ground connections.
(3) the radiation resistance R_3 of the antenna.

Item (2) can be made smaller than Item (3), and of two stations operating at the same frequency, the station having the higher radiation resistance would, in general, have the higher Item (2) resistance. That is, the two items (2) and (3) may be regarded as very roughly proportional. Let us, therefore, lump items (2) and (3), and denote their sum by

$R_0 = R_2 + R_3$. R_0 will consist mainly of the radiation resistance R_3, the expression for which is,

$$R_3 = \frac{160\pi^2 h_1^2 f^2}{s^2} \text{ ohms*} \tag{47}$$

From this expression it will be noted that the radiation resistance of the antenna is proportional to the square of the height of the capacity area.

Let us first compare the ultimate values to which the current and voltage build up, and the rate at which they build up to these values, in two receiving stations in which the only resistance is the resistance R_0. Let station A have the capacity area mounted at the height (h_1) and station B at the height (nh_1), and let the antennae be of the same area in the two stations. The constants of the two circuits are as shown in Table III.

TABLE III

Relative Constants of Low and High Antennae

	Station A	Station B
Height of capacity area	h_1	nh_1
Induced voltage (peak)	E	nE
Capacity	C	$\frac{C}{n}$
Inductance	L	nL
Radiation resistance	R	n^2R
Final current (peak)	$\frac{E}{R}$	$\frac{E}{nR}$
Final condenser voltage (peak)	$\frac{E}{RC\omega}$	$\frac{E}{RC\omega}$
Final energy stored $\frac{1}{2}LI^2$	$\frac{LE^2}{2R^2}$	$\frac{LE^2}{2nR^2}$
or $\frac{1}{2}CE^2$	$\frac{E^2}{2R^2C\omega^2}$	$\frac{E^2}{2R^2nC\omega^2}$
Final rate of radiation	$\frac{E^2}{R}$	$\frac{E^2}{R}$
Time constant	$\frac{2L}{R}$	$\frac{2L}{nR}$

*See Appendix B.

It is seen that the receiving properties of the two antennae correspond with their relative radiating properties. The current in the low antenna builds up to (n) times the current in the high; both build up to the same condenser voltage; therefore, the low antenna stores (n) times as much energy as the high, and has a time constant (n) times as long as that of the high antenna. The final rate of radiation is the same for both stations. That is, both antennae ultimately abstract energy at the same rate from the passing electromagnetic waves, but the high antenna abstracts energy at a greater rate than the low antenna during the initial stages (first few swings) of the oscillation. The high antenna will, therefore, respond much more readily to highly damped waves than will the low antenna. This means, of course, that when receiving undamped or slightly damped waves, the high antenna will be subject to greater interference from atmospheric disturbances than will the low antenna. To reduce this interference in the case of stations with high antenna, additional capacity is used in the "interference preventer" circuits. In other words, the low antenna is to be regarded as the equivalent of a high antenna and "interference preventer" combined.

Let us now suppose that the two stations under comparison contain receiving devices. The maximum amount of power is expended in these devices if the resistance R_1 of the utilizing devices is made equal to the resistance R_0. With the resistence R_1 so proportioned, the final rate at which energy is abstracted from the passing waves by the antenna is half as great as it would be if the utilizing devices were left out of the circuit. Of the energy so abstracted, one half is expended in the utilizing devices and one half is re-radiated. That is, in the case of both the high and the low antenna, the maximum rate at which energy can be expended in the utilizing devices is approximately equal to one quarter of the final rate of radiation with the utilizing devices cut out of the circuit. Since—as shown in Table III—this rate of radiation is the same for stations A & B, the conclusions previously drawn as to the relative receiving properties of the two stations are not altered by the insertion of utilizing devices with properly proportioned resistances.

35. Computed Value of the Received Power

If the station whose constants are given in Table II is used to emit undamped waves and if a similar station is used for receiving, then the maximum amount of power will be expended in the utilizing devices if their equivalent resistance is approximately .02 ohms. With such a resistance and with a distance of 5000 kilometers between the two stations and no absorption losses in transmission, the computed final values of the power received, current, etc. in the receiving station are given in Table IV.

Table IV

Computed value of received power with similar sending and receiving stations

Height of capacity areas	10	meters
Capacity of antenna to earth	.075	microfarads
Distance between stations	5000.	Km
Assumed sending conditions		
Voltage	100	peak Kv
Current	3800	peak amperes
Frequency	80000	cycles per sec.
Rate of radiation	80	Kw
Receiving Station		
Radiation plus wire & earth resistance	.02	ohms
Resistance of utilizing devices	.02	ohms
Induced voltage	.0076	peak volts
Final condenser voltage	5.0	peak volts
Final current	.19	peak amperes
Final expenditure in utilizing devices	.00035	watts
Inductance	53.	microhenries
Time constant	.0026	seconds
Time constant	210.	cycles

VII. SUMMARY

36. To Summarize:

1st. If an electro-magnetic radiator having a capacity area with a radius which is large in comparison with any feasible mounting elevation is operated at a given voltage and frequency, its radiation figure of merit is independent of the elevation at which the capacity area is mounted.

2nd. It is feasible to construct a radiator with the capacity area at a very moderate elevation which will have a radiation figure of merit equal to or greater than the values which are at present attained in long distance wireless stations by mounting the capacity area at a great elevation.

3rd. An elevated antenna will respond more readily to rapidly damped oscillations than will a low antenna. Both antennae (if the capacity areas are equal) will ultimately absorb the same amount of power from sustained or very slightly damped trains of waves. Therefore, when receiving sustained oscillations, the low antenna may be less subject to interference from atmospheric disturbances and other stations.

4th. The relative advantages of low versus high antennae for high power radio telegraph stations have been tabulated below.

Advantages or Merits of Low Versus High Antennae for High Power Stations

Low Antenna	High Antenna
Lower first cost (except where the cost of land per acre is very high) Power condensers are unnecessary Single frequency of oscillation. Apparently possible to obtain a smaller decrement than where power condensers are necessary. Less likelihood of damage by lightning. Probably less interference from "atmospheric."	Smaller antenna current—this may be of considerable advantage where arc generators or high frequency alternators are used A smaller number of insulators will be required: less likelihood of interruption due to insulator failures.

APPENDIX A

THE FORCES AT POINTS AT A GREAT DISTANCE FROM THE RADIATOR AND AT ANY ELEVATION ABOVE THE NEUTRAL PLANE

37. Let P in Fig. 18 represent any point at a great distance from the radiator, and let Q represent any small portion of the positive moving charge.

We wish to calculate the potentials and the forces at the point at the instant t which arise from this moving charge and its image $-Q$.

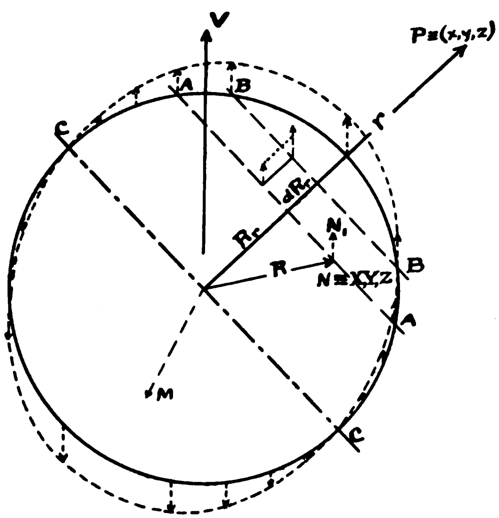

FIG. 18

Let some element in the moving charge Q be selected, and let us suppose that this element reaches the point O in space at such an instant (t_o) previous to (t) that the distance OP will, at the velocity of propogation (s) in the intervening medium, be traversed in the interval $(t - t_o)$. Then

$$OP \text{ or } r = s\,(t - t_o) \qquad (48)$$

Let O (regarded as a point fixed in space) be chosen as the origin of coordinates and let the coordinates of P be represented by x, y, z.

Let us take any other element of the charge, as the element N, whose coordinates at the instant (t_o) are X, Y, Z, and determine its coordinates when it is occupying such a position that the influence propogated from the element N will reach the point P at the instant (t), that is, simultaneously with the influence from the element at O. This position will be called "the effective position of the element N corresponding to the instant (t)". Let this effective position be represented by N_1 ($= X_1, Y_1, Z_1$.)

Assume that this effective position N_1 is reached at an instant (t_1) which is later than the instant (t_o) by the interval τ.

Then

$$t_1 = t_o + \tau \qquad (49)$$

The condition that N_1 shall be the effective position of the element is that the distance

$$N_1P \text{ shall equal } s(t - t_1) = s(t - t_o - \tau) \qquad (50)$$

If the charge Q is assumed to be all moving in the same direction with the velocity **V** (components V_1, V_2, and V_3), then

$$\left.\begin{aligned} X_1 &= X + V_1\tau \\ Y_1 &= Y + V_2\tau \\ Z_1 &= Z + V_3\tau \end{aligned}\right\} \qquad (51)$$

But

$$N_1P = \sqrt{(x - X_1)^2 + (y - Y_1)^2 + (z - Z_1)^2}$$

$$= \sqrt{(x - X - V_1\tau)^2 + (y - Y - V_2\tau)^2 + (z - Z - V_3\tau)^2} \qquad (52)$$

Therefore, from (50) and (52),

$$s^2(t - t_o - \tau)^2 = (x - X - V_1\tau)^2 + (y - Y - V_2\tau)^2 + (z - Z - V_3\tau)^2 \qquad (53)$$

Expanding, and dropping infinitesimals of higher orders.

$$\tau\left(s - \frac{xV_1 + yV_2 + zV_3}{r}\right) = \frac{xX + yY + zZ}{r}$$

But $\frac{xV_1+yV_2+zV_3}{r}$ is the component of the velocity of the charge in the direction OP, or along the radius (r) to the point P.

$$\text{Representing} \frac{xV_1+yV_2+zV_3}{r} \text{ by } V_r, \tag{54}$$

$$\tau = \frac{1}{s-V_r} \frac{xX+yY+zZ}{r} \tag{55}$$

It will be noted that $\frac{xX+yY+zZ}{r}$ is the projection of the radius ON on the radius OP.

$$\text{Representing} \frac{xX+yY+zZ}{r} \text{ by } R_r, \tag{56}$$

equation (55) may be written

$$\tau = \frac{R_r}{s-V_r} \tag{57}$$

The element which at the instant (t_o) has the coordinates N_1 (= X, Y, Z,) has its effective position corresponding to the instant (t) at the point $N_1 = X_1, Y_1, Z_1$, and its distance from P, namely N_1P is (by equations (50) and (57),

$$N_1P = s(t - t_o - \tau) = r - s\,\tau = r - \frac{sR_r}{s-V_r} \tag{58}$$

38. Let us now assume that the moving charge Q is spherical, that the element O is the center of the charge, and that the integrations indicated in equations (26) and (27) for the retarded potentials Φ and $\mathbf{A}$ are to be carried out over this charge for the instant (t). It will be noted that the volume occupied by the charge in its "effective position corresponding to the instant (t)" (that is, the volume over which the integrations are to be taken) is not spherical in shape. This is due to the fact that the influence of any element of the charge, as N, which is nearer to the point P than is the center O of the sphere, reaches P before the influence from O. The effective position of N is therefore some position, as N_1,

which it occupies later in time. That is, the effective position of any element whose distance from P is less than OP will be found by displacing the element in the direction of movement of the charge by an amount which in the case of slowly moving charges will be shown to be proportional to the distance R_r of the element from the plane CC, and to the velocity of the moving charge. On the other hand, the effective position of any element M whose distance from P is greater than OP will be found by displacing the element in a direction opposite to the direction of movement of the charge. The integrations indicated in equations (26) and (27) for the retarded potentials must, therefore, be carried on over the space found by distorting the sphere as indicated by the dotted outlines of Fig. 18.

The relative volume of the spherical space and the space within the dotted outline may be found as follows. Consider a differential slice of the charge included between two planes AA and BB perpendicular to the radius OP. Let the projection on OP of the radius to any point in the plane AA be R_r, and the projection of any point in BB be $R_r + dR_r$. The effective position of any element in the AA plane is given by equations (51) and (57) as

$$X_1 = X + V_1 \tau = X + \frac{R_r}{s - V_r} V_1$$

$$Y_1 = Y + V_2 \tau = Y + \frac{R_r}{s - V_r} V_2$$

$$Z_1 = Z + V_3 \tau = Z + \frac{R_r}{s - V_r} V_3$$

That is, in their effective positions corresponding to the instant (t) all elements in the AA plane are displaced in the direction of movement of the charge by the amount $\frac{V}{s - V_r} R_r$ In like manner, all elements in the BB plane are displaced by the amount $\frac{V}{s - V_r} (R_r + dR_r)$

The displacement of the BB plane is therefore greater than that of the AA plane by the amount $\frac{V}{s - V_r} dR_r$. Since this difference is between displacements in the direction of motion of the charge, the *increase* in the *perpendicular* distance between the planes is $\frac{V_r}{s-V_r} dR_r$

Hence the volume between the planes in their effective positions is equal to $\left(1+\frac{V_r}{s-V_r}\right) = \left(1+\frac{V_r}{s} - \frac{V_r^2}{s^2}\ldots\ldots\right)$ times the volume between the planes AA and BB. That is, the effective volume over which the integration is to be taken is greater than the spherical volume in the ratio of $\left(1+\frac{V_r}{s} - \frac{V_r^2}{s^2}\ldots\ldots\right)$ to 1.

39. For the case of slowly moving charges, as in the low frequency station whose constants are discussed in *Section 31*, we may now repeat the arguments used in *Sections 22 to 24*, save that the charge must be multiplied by $\left(1+\frac{V_r}{s} - \frac{V_r^2}{s^2}\ldots\right)$ and that P must be taken to represent a point at any elevation above the neutral plane and at a great distance from the radiator.

Letting θ represent the angle between the line OP and the axis of the radiator, the following expressions are obtained for the retarded potentials at P.

$$\Phi = \frac{Qh_o}{2\pi r}\left[-\frac{\omega\cos\theta}{s}\sin\omega\left(t-\frac{r}{s}\right) + \frac{\cos\theta}{r}\cos\omega\left(t-\frac{r}{s}\right)\right] \qquad (59)$$

$$\mathbf{A} = A_2 = -\frac{Q\,h_o f}{r}\sin\omega\left(t-\frac{r}{s}\right) \qquad (60)$$

From these expressions have been omitted the terms which in the case of slowly moving charges, are negligible at great distances from the radiator.

From these expressions for the retarded potentials, the following expressions may be derived for the *radiant* terms in the electric and magnetic force at P.

$$H_1 = H_2 = 0$$

$$H_3 = \frac{2\pi Q h_0 f^2}{sr} \sin\theta \cos\omega\left(t - \frac{r}{s}\right) \tag{61}$$

$$F_1 = -\frac{240\pi^2 Q h_0 f^2}{sr} \cos\theta \sin\theta \cos\omega\left(t - \frac{r}{s}\right) \tag{62}$$

$$F_2 = \frac{240\pi^2 Q h_0 f^2}{sr} \sin^2\theta \cos\omega\left(t - \frac{r}{s}\right) \tag{63}$$

$$F_3 = 0$$

The resultant electric force (the resultant of F_1 and F_2) is seen to be normal to the radius OP, to have the absolute value F given below, and to lie in the plane determined by the point P and the axis of the radiator.

$$F = \frac{240\pi^2 Q h_0 f^2}{sr} \sin\theta \cos\omega\left(t - \frac{r}{s}\right) \tag{64}$$

Since the magnetic force is perpendicular to this plane, the electric and magnetic forces are both perpendicular to the line OP, and they are perpendicular to each other. Therefore, the radiant vector is in the direction of the line OP, and the rate P_1 at which energy streams out across a plane perpendicular to OP at the point P is

$$P_1 = \frac{480\pi^3}{s^2} \frac{Q^2 h_0{}^2 f^4}{r^2} \sin^2\theta \cos^2\omega\left(t - \frac{r}{s}\right) \text{ (watts per sq. cm.)} \tag{65}$$

APPENDIX B

THE RATE OF RADIATION AND THE RADIATION RESISTANCE

40. Rate of Radiation

At any point at a great distance from the radiator, the direction of flow of energy is normal to the surface of a sphere passing through the point and having the radiator as a center. The rate P_1 at which energy passes outward across a square centimeter of the surface of the sphere at the point is given in equation (65). The rate of radiation is a maximum in the equatorial plane of the radiator and falls off toward the pole as the square of the sine of the angular distance of the point from the pole.

The *mean* rate of radiation P from the radiator (with undamped oscillations) will be found by integrating over the entire surface of a hemisphere described about the radiator, and then writing the mean value for a cycle of the expression so obtained. Upon carrying out this integration, the following expression is obtained for the rate of radiation P through a hemisphere described about the radiator.

$$P = \frac{320\pi^4 Q^2 h_0^2 f^4}{s^2} \text{ watts} \qquad (66)$$

41. Radiation Resistance

The charges depicted in Fig. 5 *each* cross the neutral plane 2f times per second. The quantity of electricity which crosses the neutral plane per second is, therefore, 4fQ coulombs, or the moving charges convey the same quantity as an alternating current whose average value is 4fQ amperes, and whose r. m. s. value (I) is

$$I = \sqrt{2}\,\pi f Q \quad \text{r. m. s. amperes} \qquad (67)$$

Now the "radiation resistance" may be defined as a fictitious resistance of such a value that the product of the radiation resistance times the square of the current will equal the rate of radiation from the radiator. That is, the radiation resistance R_3 is defined by the equation

$$R_3 = \frac{P}{I^2}$$

Substituting the values of P and I as given in equations (66) and (67),

$$R_3 = \frac{160\,\pi^2 f^2 h_o^2}{s^2} \qquad (68)$$

42. Logarithmic Decrement Due to the Radiation of Energy

The logarithmic decrement (δ) of the oscillation is defined as the Naperian logarithm of the ratio of any peak value to that following it by a cycle.

Let it be assumed that the oscillation of the radiator is not sustained but is damped, and that the only loss is that due to radiation. For a slightly damped circuit containing resistance, inductance and capacity in series, the value of the logarithmic decrement is given quite accurately by the expression

$$\delta = \frac{2\pi\, R}{R_c} \qquad (69)$$

in which,

R is the actual resistance of the circuit, and

R_c is its critical resistance, defined by the equation

$$R_c = 2\sqrt{\frac{L}{C}} = \frac{2}{2\pi f C}$$

where f is the natural frequency of oscillation of the circuit.

Now, in the case of a low extended antenna of area A, the capacity C is given approximately by the expression.

$$C = \frac{pA}{h_o}$$

Whence

$$R_c = \frac{120\ s\ h_o}{fA} \tag{70}$$

Substituting in equation (69) the value of R_c as given in equation (70) and for R the value of R_3 as given in equation (68), the following expression is obtained for the logarithmic decrement.

$$\delta = \frac{8\pi^3\ f^3\ h_o A}{3\ s^3} \tag{71}$$